DAC Spectrum with Output Clock Jitter

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The output power spectrum of a DAC with nonuniformly spaced output transitions is calculated exactly. The result is interpreted and sample spectra are plotted. The relation of the spectral method with time domain analysis is discussed.

Notation and Assumptions

1^z A shorthand notation for any complex number z:

$$1^z = e^{2\pi i z} \tag{1}$$

f_{clk} Nominal output clock frequency.

T_{clk} $1 / f_{clk}$. Clock period.

N A specified integer setting the total measurement time, T.

T NT_{clk}, measurement time.

t Time. The DAC output is a function of this continuous parameter.

t_n Time of DAC output transition n, from plateau level $x_{in,n-1}$ to $x_{in,n}$, due to the n^{th} clock transition, n integer. We assume that $x_{in,n+N} = x_{in,n}$. The measurement time window is $[t_0, t_N[$.See also $x_{in,n}$ and θ_n.

f_{bin} Bin size in frequency space, equal to $1/T = f_{clk}/N$.

f Frequency, only values $f = p\, f_{bin}$, p integer, are considered here.

f_{in} Frequency of ideal input signal. It is assumed that $f_{in} = p_{in}\, f_{bin}$, p_{in} integer.

ν $f / f_{clk} = fT_{clk}$. Normalized frequency.

Δt_n Error of t_n from ideal nT_{clk} relative to T_{clk}, equal to $t_n - nT_{clk}$.

$\theta(t)$ Phase error of a sinusoidal clock signal generating the transition times $\theta_n = \theta(nT_{clk})$.

θ_n Phase error of t_n from ideal nT relative to T_{clk}, equal to $2\pi\Delta t_n/T_{clk}$, or

$$t_n = \left(n + \frac{\theta_n}{2\pi}\right)T_{clk} \tag{2}$$

The θ_n are assumed to be identically zero-mean normally distributed, but *not* independent. The process θ is assumed to be wide-sense stationary (WSS), i.e. the mean, $E\{\theta_m\}$, and the covariance, $E\{\theta_m\theta_{m+n}\}$, are independent of m [1]. In our case $E\{\theta_m\} = 0$. We write for the covariance

$$R_{\theta,n} = E\{\theta_m\theta_{m+n}\} \qquad \text{“Covariance” } (\theta\,\text{WSS}) \tag{3}$$

θ_f^D DFT of θ_n, a function of frequency, f. For our DFT convention, see x_f^D below.

θ_f^S SFT of $\theta(t)$, a function of frequency, f. For our SFT convention, see x_f^F below.

u_n Error of t_n from ideal nT_{clk} relative to T_{clk}, equal to $\Delta t_n/T_{clk} = \theta_n/2\pi$, or

$$t_n = (n + u_n)T_{clk} \tag{4}$$

Since u is proportional to θ, it is evidently also WSS

A Amplitude of the sinusoidal wave (5) from which the DAC levels (6) are derived.

$x_{in}(t)$ DAC input signal reference as a continuous function of time, t:

$$x_{in}(t) = A\cos(2\pi f_{in} t) \tag{5}$$

$x(t)$ DAC output signal as a continuous function of time, t.

$x_{in,n}$ Value of sinusoidal wave sampled uniformly with period T_{clk}:

$$x_{in,n} = x_{in}(nT_{clk}) = A\cos(2\pi f_{in} nT_{clk})\,,\ n \text{ integer} \tag{6}$$

The $x_{in.n}$, $n = 0, 1, \ldots, N - 1$, are the DAC output plateaus during $[t_n, t_{n+1}[$. The DAC levels are ideal, and only the output transition times, t_n, are non-ideal [2]. We ignore DAC level quantization, i.e. the DAC has infinite resolution.

x_f^D Discrete Fourier transform (DFT) of a discrete function x_n, a function of discrete frequency, f. The DFT pair is [3][4]:

$$x_n = \sum_{f:f_{bin}=0}^{f_{clk}-f_{bin}} x_f^D 1^{fnT_{clk}}\,,\ n = 0, 1, \ldots, N-1 \tag{7}$$

$$x_f^D = \frac{1}{N}\sum_{n=0}^{N-1} x_n\, 1^{-fnT_{clk}}\,,\ f = p\, f_{bin},\ p \text{ any integer} \tag{8}$$

Note that we allow f outside the interval $[0, f_{clk}]$, so the DFT repeats with period f_{clk} [5]. For a sinusoidal wave (6):

$$x_{in,f}^D = \frac{A}{2}\sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} \tag{9}$$

x_f^S Series Fourier transform (SFT) of a continuous function $x(t)$, a function of discrete frequency, f. The SFT pair is [4]

$$x(t) = \sum_{f:f_{bin}=-\infty}^{\infty} x_f^S 1^{ft}\,,\ t \in [0, T[\tag{10}$$

$$x_f^S = \frac{1}{T}\int_0^T x(t) 1^{-ft} dt\,,\ f = p\, f_{bin},\ p \text{ any integer} \tag{11}$$

The SFT does not repeat with period f_{clk}: Components with arbitrarily large f in (10) add to the description of the continuous function $x(t)$ [6]. For a sinusoidal wave (5):

$$x_{in,f}^S = \frac{A}{2}\sum_{s=\pm 1}\delta_{f,sf_{in}} \tag{12}$$

Tools

Here we present some useful equations. They are easy to prove from the definitions above [7]. We use these in the next section to derive our main result. In this section z refers to a general random variable, which may be the specific x, or θ, or u, etc. defined earlier.

Aliasing relationship between DFT and SFT:

$$z_f^D = \sum_{k=-\infty}^{\infty} z_{f+kf_{clk}}^S \qquad \text{“Aliasing Theorem”} \tag{13}$$

A trivial example of (13) is provided by (9) and (12).

Parseval’s theorem for DFT and SFT [8]:

$$\frac{1}{N}\sum_{n=0}^{N-1} |z_n|^2 = \sum_{f:f_{bin}=0}^{f_{clk}-f_{bin}} |z_f^D|^2 \qquad \text{“Parseval’s Theorem” (DFT)} \tag{14}$$

$$\frac{1}{T}\int_0^T z^2(t)dt = \sum_{f:f_{bin}=-\infty}^{\infty} |z_f^S|^2 \qquad \text{“Parseval’s Theorem” (SFT)} \tag{15}$$

More generally, the single-measurement (non-ensemble-averaged) autocorrelations are [9]:

$$c_{z,n} = \frac{1}{N}\sum_{m=0}^{N-1} z_m^* z_{m+n} = \sum_{f:f_{bin}=0}^{f_{clk}-f_{bin}} |z_f^D|^2\, 1^{nfT_{clk}} \tag{16}$$

$$c_z(t) = \frac{1}{T}\int_0^T z^*(t')z(t'+t)dt' = \sum_{f:f_{bin}=-\infty}^{\infty} |z_f^S|^2 1^{ft} \tag{17}$$

Fourier transforming both of these gives

$$c_{z,f}^{D\,(S)} = |z_f^{D\,(S)}|^2 \qquad \text{“Wiener-Khinchin Theorem I”} \tag{18}$$

Although the z may not, and in our case often will not, be WSS, we can still define the expected correlation functions, C_z, as

$$C_{z,n} = E\{c_{z,n}\} = \frac{1}{N}\sum_{m=0}^{N-1} E\{z_m^* z_{m+n}\} \tag{19}$$

$$C_z(t) = E\{c_z(t)\} = \frac{1}{T}\int_0^T E\{z^*(t')z(t'+t)\}dt' \tag{20}$$

Taking the expectation value of (18) gives Fourier transforms of the ensemble-averaged autocorrelation functions in terms of the Fourier transforms themselves:

$$C_{z,f}^{D\,(S)} = E\{|z_f^{D\,(S)}|^2\} \qquad \text{“Wiener Khinchin Theorem II”} \tag{21}$$

According to (18) and (21) the Fourier transforms of autocorrelation functions are real and invariant under frequency inversion (replacement f by $-f$).

If z is WSS, autocorrelation equals covariance:

$$C_{z,n} = R_{z,n} \qquad (z \text{ WSS}) \tag{22}$$

$$C_z(t) = R_z(t) \qquad (z \text{ WSS}) \tag{23}$$

For example, (22) applies if the process z is either θ or u, but not x, defined in the previous section.

From (21), (22), (23) we find a third version of the Wiener Khinchin theorem for WSS z

$$C_{z,f}^{D\,(S)} = E\{|\, z_f^{D\,(S)}\,|^2\} \qquad \text{“Wiener Khinchin Theorem III” } (z \text{ WSS}) \tag{24}$$

Spectrum Derivation

A. Amplitude Spectrum

The DAC output is

$$x(t) = \sum_{n=0}^{N-1} B(t_n, t_{n+1}; t) x_{in,n} \tag{25}$$

where the "block" function B is given by

$$B(a,b;t) = \begin{cases} 0 \text{ if } t \notin [a,b[\\ 1 \text{ if } t \in [a,b[\end{cases} \text{, assumes } a < b \tag{26}$$

see Figure 1. We will only assume in sub-section C that $x(t)$ is sinusoidal.

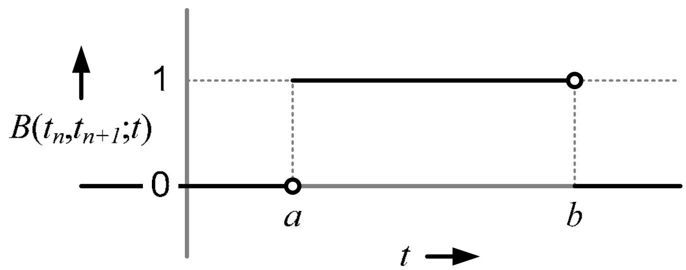

Figure 1. Function $B(a, b; t)$ of t for given $a < b$.

The SFT of B with respect to t is

$$\left[B(t_n, t_{n+1}; t)\right]_f^{St} = \frac{1}{T}\int_0^T B(t_n, t_{n+1}; t) 1^{-ft}\, dt = \frac{1^{-ft_n} - 1^{-ft_{n+1}}}{2\pi i f T} \tag{27}$$

The SFT of the DAC output follows from (25) and (27):

$$x_f^S = \frac{1}{T}\int_0^T B(t_n, t_{n+1}; t)\, 1^{-ft}\, dt = \frac{1}{2\pi i f T}\sum_{n=0}^{\infty} x_{in,n}(1^{-ft_n} - 1^{-ft_{n+1}}) \tag{28}$$

Then with (4) and $T = NT_{clk}$

$$x_f^S = \frac{1}{2\pi i\, \nu N}\sum_{n=0}^{\infty} x_{in,n} 1^{-\nu n}\left(1^{-\nu u_n} - 1^{-\nu(1+u_{n+1})}\right) \tag{29}$$

where

$$\nu = fT_{clk} = \frac{f}{f_{clk}} \qquad \text{"Normalized Frequency"} \tag{30}$$

If we assume

$$|\nu u_n| = \left|fT_{clk}u_n\right| = \frac{|f||u_n|}{f_{clk}} << 1 \tag{31}$$

then to first order in the lhs of (31) the SFT (29) evaluates as

$$x_f^S = 1^{-\nu/2}\mathrm{sinc}(\pi\nu)x_{in,f}^D + \frac{1}{N}\sum_{n=0}^{\infty}(x_{in,n} - x_{in,n-1})\,1^{-\nu n}u_n \tag{32}$$

But in our main derivation we postpone assumption (31) until sub-section D. For the noise free case, $u_n = 0$, we will use a subscript "0," and both (29) or quicker (32) show

$$\begin{aligned} x_{0,f}^D &\overset{(13)}{=} \sum_{k=-\infty}^{\infty} x_{0,f+kf_{clk}}^S \overset{(28)}{=} x_{in,f}^D \sum_{k=-\infty}^{\infty} 1^{-(\nu+k)/2}\mathrm{sinc}(\pi(\nu+k)) = \\ &= 1^{-\nu/2} x_{in,f}^D \frac{\sin(\pi\nu)}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{\nu+k} \overset{(61)}{=} (1-1^{-\nu})x_{in,f}^D \end{aligned} \tag{33}$$

For the last equality, see (73) in Appendix A. The peculiar relationship between the input and output DFTs reflects the fact that $x_{0,n}$ as derived from its SFT is the average of $x_{in,n}$ and $x_{in,n-1}$. Equivalently, performing a DFT plus inverse DFT on $B(a, b; t)$ changes the values at a and b to the average of 0 and 1, or ½. This averaging effect at discontinuities is inherent in Fourier transforms.

B. Power Spectrum

From (29) we have

$$E\{|x_f^S|^2\} = \sum_{n,n'=0}^{\infty} x_{in,n}^* x_{in,n'} g_{\nu,n-n'} \tag{34}$$

where f and ν are related through (30) and

$$g_{\nu,n-n'} = \frac{1^{\nu(n-n')}}{(2\pi\nu N)^2} E\{1^{\nu(u_n-u_{n'})} - 1^{\nu(u_n-1-u_{n'+1})} - 1^{\nu(1+u_{n+1}-u_{n'})} + 1^{\nu(u_{n+1}-u_{n'+1})}\} \tag{35}$$

which is known to only depend on $n – n'$ because u is WSS. We rewrite (34) as

$$E\{|x_f^S|^2\} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{in,n}^* x_{in,n+m} g_{\nu,-m} = N\sum_{m=0}^{\infty} c_{x_{in},m} g_{\nu,-m} = N\sum_{m=0}^{\infty}\sum_{f':f_{bin}=0}^{f_{clk}-f_{bin}} |x_{in,f'}^D|^2\, 1^{\nu' m} g_{\nu,-m} \tag{36}$$

For the identically normally distributed u_n we have according to Appendix B, (77)

$$E\{1^{\nu(u_n-u_{n'})}\} = e^{-(2\pi\nu)^2(\sigma_u^2 - R_{u,n-n'})} = e^{-\nu^2(\sigma_\theta^2 - R_{\theta,n-n'})} \tag{37}$$

So (35) evaluates to

$$g_{\nu,-m} = \frac{1^{\nu m}}{(2\pi\nu N)^2} e^{-\nu^2\sigma_\theta^2}\left(2e^{(2\pi\nu)^2 R_{\theta,m}} - 1^{-\nu} e^{(2\pi\nu)^2 R_{\theta,m+1}} - 1^{\nu} e^{(2\pi\nu)^2 R_{\theta,m-1}}\right) \tag{38}$$

and (36) gives

$$\begin{aligned} E\{|x_f^S|^2\} &= \frac{1}{(2\pi\nu)^2 N} e^{-\nu^2\sigma_\theta^2}\sum_{m=0}^{\infty}\sum_{f':f_{bin}=0}^{f_{clk}-f_{bin}} |x_{in,f'}^D|^2\, 1^{(\nu'-\nu)m} e^{\nu^2 R_{\theta,m}}(2 - 1^{-\nu'} - 1^{\nu'}) = \\ &= \frac{1}{(2\pi\nu)^2 N}\sum_{m=0}^{N-1} e^{-\nu^2(\sigma_\theta^2 - R_{\theta,m})}\sum_{f':f_{bin}=0}^{f_{clk}-f_{bin}} |x_{in,f'}^D|^2\, 4\sin^2(\pi\nu')\, 1^{(\nu'-\nu)m} \end{aligned} \tag{39}$$

This may also be written as

$$E\{|x_f^S|^2\} = \frac{1}{(2\pi\nu)^2 N} \sum_{m=0}^{N-1} 1^{-\nu m} e^{-\nu^2(\sigma_\theta^2 - R_{\theta,m})} (2C_{x_{in},m} - C_{x_{in},m-1} - C_{x_{in},m+1}) \tag{40}$$

C. Sinusoidal Input

Assume $x_{in,n}$ is derived from a sinusoidal wave as indicated in (5) and (6), i.e.

$$x_{in,n} = \frac{A}{2} \sum_{s=\pm 1} 1^{s\nu_{in} n} \tag{41}$$

$$x_{in,f}^D = \frac{A}{2} \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f, sf_{in}+kf_{clk}} \tag{42}$$

Substitution into (39) gives

$$E\{|x_f^S|^2\} = S_{in} A^2 \frac{\sin^2(\pi\nu_{in})}{(2\pi\nu)^2} \frac{1}{N} \sum_{\substack{m=0 \\ s=\pm 1}}^{N-1} 1^{(s\nu_{in}-\nu)m} e^{-\nu^2(\sigma_\theta^2 - R_{\theta,m})} \tag{43}$$

where the factor S_{in} is 1, except when f_{in} is an integer multiple of the Nyquist frequency, $f_{clk}/2$, and then it is given by (86) or (90) in Appendix C.

As a check of (43) we note that the SFT of the noise free output $x_0(t)$ is, from (32) and (9)

$$x_{0,f}^S = 1^{-\nu/2} \mathrm{sinc}(\pi\nu) x_{in,f}^D \overset{(9)}{=} \frac{A}{2} 1^{-\nu/2} \mathrm{sinc}(\pi\nu) \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f, sf_{in}+kf_{clk}} \tag{44}$$

so its absolute value squared is

$$|x_{0,f}^S|^2 = S_{in} \left(\frac{A}{2}\right)^2 \mathrm{sinc}^2(\pi\nu) \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f, sf_{in}+kf_{clk}} \tag{45}$$

which is indeed the no-noise case of (43).

D. Small Jitter Approximation

The exact equation (43) can not be simplified further without knowing more about the autocorrelation $R_{r,m}$. But often σ_r is relatively small, i.e. (30) holds, or equivalently

$$|\nu|\sigma_r = \frac{|f|\sigma_r}{f_{clk}} = |f| T_{clk} \sigma_r = |f| \sigma_{\Delta t} << 1 \qquad \text{"Small Jitter"} \tag{46}$$

where $\sigma_{\Delta t}$ is the standard deviation of the timing errors Δt_n, or jitter. The validity of (46) depends on the particular frequency f we are looking at in the spectrum. For any given $\sigma_{\Delta t}$ there will be frequencies f for which (46) fails badly. We come back to this point in the final two sections.

Assuming (46) we expand the last exponential in (43) to first order to obtain

$$E\{|x_f^S|^2\} \approx (1-(\nu\sigma_\theta)^2)|x_{0,f}^S|^2 + S_{in}A^2\sin^2(\pi\nu_{in})\sum_{s=\pm1} R_{u,f-sf_{in}}^D \approx \\ \approx |x_{0,f}^S|^2 + S_{in}A^2\sin^2(\pi\nu_{in})\sum_{s=\pm1} R_{u,f-sf_{in}}^D \tag{47}$$

where we used (46) another time to simplify the first term. Since we will stay within the small jitter approximation for most of this paper, we will use regular equality signs from here on.

E. Equivalent Forms

Using both the aliasing theorem and the Wiener-Khinchin theorem we can cast in a number of equivalent forms, which we simply list here. For the normalized frequency ν see (30). The noise-free spectral power $|x_{0,f}^S|^2$ is given by (45).

$$E\{|x_f^S|^2\} = |x_{0,f}^S|^2 + S_{in}A^2\sin^2(\pi\nu_{in})\sum_{s=\pm1} E\{|u_{f-sf_{in}}^D|^2\} \tag{48}$$

$$E\{|x_f^S|^2\} = |x_{0,f}^S|^2 + S_{in}\left(\frac{A}{2}\right)^2 \frac{\sin^2(\pi\nu_{in})}{\pi^2}\sum_{s=\pm1} E\{|\theta_{f-sf_{in}}^D|^2\} \tag{49}$$

$$E\{|x_f^S|^2\} = |x_{0,f}^S|^2 + S_{in}\left(\frac{A}{2}\right)^2 \frac{\sin^2(\pi\nu_{in})}{\pi^2}\sum_{\substack{k=-\infty \\ s=\pm1}}^{\infty} E\{|\theta_{f-(sf_{in}+kf_{clk})}^S|^2\} \tag{50}$$

$$E\{|x_f^S|^2\} = S_{in}\left(\frac{A}{2}\right)^2 \sum_{\substack{k=-\infty \\ s=\pm1}}^{\infty} \left(\operatorname{sinc}^2(\pi\nu)\delta_{f,sf_{in}+kf_{clk}} + \frac{\sin^2(\pi\nu_{in})}{\pi^2} E\{|\theta_{f-(sf_{in}+kf_{clk})}^S|^2\}\right) \tag{51}$$

$$E\{|x_f^S|^2\} = S_{in}\left(\frac{A}{2\pi}\right)^2 \sin^2(\pi\nu_{in}) \sum_{\substack{k=-\infty \\ s=\pm1}}^{\infty} \left(\frac{\delta_{f,sf_{in}+kf_{clk}}}{\nu^2} + E\{|\theta_{f-(sf_{in}+kf_{clk})}^S|^2\}\right) \tag{52}$$

$$E\{|x_f^S|^2\} = S_{in}\left(\frac{A}{2\pi}\right)^2 \sin^2(\pi\nu_{in}) \sum_{\substack{k=-\infty \\ s=\pm1}}^{\infty} \left(\frac{\delta_{f,sf_{in}+kf_{clk}}}{(s\nu_{in}+k)^2} + E\{|\theta_{f-(sf_{in}+kf_{clk})}^S|^2\}\right) \tag{53}$$

Main Result

We rewrite (51) in terms of primitive variables, leaving off S_{in}

$$E\{|x_f^S|^2\} = \left(\frac{A}{2}\right)^2 \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \left(\text{sinc}^2\left(\frac{\pi f}{f_{clk}}\right) \delta_{f, sf_{in}+kf_{clk}} + \frac{1}{\pi^2}\sin^2\left(\frac{\pi f_{in}}{f_{clk}}\right) E\{|\theta^S_{f-(sf_{in}+kf_{clk})}|^2\} \right) \tag{54}$$

We summarize the assumptions:

1. The timing errors are identically normally distributed (but not independent). Equivalent to phase errors being normally distributed (but not a white phase noise spectrum)
2. Time invariance of the timing error (or time domain phase noise) correlations, see (3)
3. Input is a sinusoidal wave, single-tone, see (5), (6)
4. Infinite DAC level resolution, or no quantization error. This is implicit in (5), (6)
5. Small jitter: $|f|\sigma_{\Delta t} << 1$, see (46)
6. Input frequency, f_{in}, is not an integer multiple of the Nyquist frequency, f_{clk} / 2. Otherwise (54) must be multiplied by S_{in}, given by (86), or more generally by (90)

We did *not* have to assume that the cross-correlation between the signal and the noise is zero. Some authors implicitly assume this to be the case without justification [10]. Assumption 1 was critical in allowing us to avoid this issue entirely, see previous section, sub-section B. Y. C. Yenq discusses the DAC spectrum with a time varying clock [11], but derives only an aggregate signal-to-noise ratio (SNR) and not the expected power spectrum

If we also have $N >> 1$, (54) holds approximately without even taking (ensemble) expectation values.

Interpretation

From Parseval's theorem (15), we have:

$$P_x = \frac{1}{T}\int_0^T x^2(t)dt = \sum_{f:f_{bin}=-\infty}^{\infty} |x_f^S|^2 \tag{55}$$

The left hand side is interpreted as the average "power" during the measurement time T [12], and the integral, i.e. TP_x, is the total "energy." The "power spectrum" $|x_f^S|^2$ is the average power carried by the signal $x(t)$ in a bandwidth f_{bin} centered at frequency f. This is proportional to the power level displayed on a spectrum analyzer (SA) for the signal $x(t)$ at frequency f with bandwidth set to f_{bin}. With averaging selected on the SA, it emulates the expectation value $E(|x_f^S|^2)$. $|x_f^S|^2$ is called the power spectrum of x [13].

Hence $|\theta_f^S|^2$ is the power spectrum of the phase noise, or phase noise spectrum for short. With the Wiener-Khinchin theorem (18) and the fact that θ is wide-sense stationary we have

$$R^S_{\theta,f} = C^S_{\theta,f} = E\{|\theta_f^S|^2\} \tag{56}$$

where

$$C_\theta(t) = \frac{1}{T}\int_0^T E\{\theta(t')\theta(t'+t)\}dt' = E\{\theta(t')\theta(t'+t)\} = R_\theta(t) \tag{57}$$

is the time correlation function as well as covariance of $\theta(t)$.

According to (54) the non-ideal DAC output spectrum is the sum of the ideal "sinc-squared" DAC spectrum, with no clock jitter, and the phase noise spectrum shifted by $\pm f_{in}$, including all f_{clk} aliased components, multiplied by $(\sin(\pi f_{in}/f_{clk})/\pi)^2$. The aliasing of the phase noise does not contribute significantly for close-in phase noise, but it matters for white phase noise. The same is true for the effects of sampling clock phase noise on an ADC spectrum [14].

Plots

To generate sample plots, we need a sample phase noise spectrum. Through the Wiener-Khinchin theorem (21) such a spectrum is the SFT of the continuous-time phase correlation function, $C_\theta(t)$, defined as in (20). The simplest possible time-domain expected correlation form, beyond the trivial uncorrelated one, is proportional to $e^{-|t|/\tau}$, where τ is the correlation time of the underlying physical process [15]. But this is not yet periodic in T. For our formalism to be exact we artificially make this expected correlation function periodic. It can be shown that for large T, the error is negligible. We suppress the rather lengthy derivations for some of the following equations in this version of the manuscript.

Continuous-time expected phase correlation function

$$C_\theta(t) = \sigma_\theta^2 \frac{e^{-t/\tau} + e^{(t-T)/\tau}}{1+e^{-T/\tau}},\ t \in [0,T],\ \text{repeats every } T \tag{58}$$

Discrete transition time expected correlation function

$$C_{\theta,n} = C_\theta(nT_{clk}) = \sigma_\theta^2 \frac{e^{-nT_{clk}/\tau} + e^{(n-N)T_{clk}/\tau}}{1+e^{-T/\tau}},\ n = 0,1,\ldots,N,\ \text{repeats every } N \tag{59}$$

SFT of $C_\theta(t)$ (and remember that f = integer x f_{in})

$$E\{|\theta_f^S|^2\} = C_{\theta,f}^S = \frac{\sigma_\theta^2}{T}\frac{1-e^{-T/\tau}}{1+e^{-T/\tau}}\frac{2\tau}{1+(2\pi f\tau)^2} \tag{60}$$

DFT of $C_{\theta,n}$

$$E\{|\theta_f^D|^2\} = C_{\theta,f}^D = \frac{\sigma_\theta^2}{N}\frac{1-e^{-T/\tau}}{1+e^{-T/\tau}}\frac{1-e^{-2T_{clk}/\tau}}{1-2e^{-T_{clk}/\tau}\cos(2\pi f T_{clk}) + e^{-2T_{clk}/\tau}} \tag{61}$$

The first equality in (60) and (61) are FFT and DFT versions of the Wiener-Khinchin Theorem II (21). Note that the DFT is periodic in frequency, f, with period f_{clk}, as required, while the SFT is not.

Sum rules

$$\sum_{f:f_{bin}=-\infty}^{\infty} C_{\theta,f}^S = \sigma_\theta^2 \tag{62}$$

$$\sum_{f:f_{bin}=0}^{f_{clk}-f_{bin}} C^D_{\theta,f} = C_{\theta,0} = \sigma^2_\theta \tag{63}$$

We will use the form (49) for the output spectrum, dropping the S_{in} as we did in (54)

$$\boxed{E\{|x^S_f|^2\} = \left(\frac{A}{2}\right)^2 \left(\operatorname{sinc}^2\left(\frac{\pi f}{f_{clk}}\right)\sum_{\substack{k=-\infty\\ s=\pm1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} + \frac{1}{\pi^2}\sin^2\left(\frac{\pi f_{in}}{f_{clk}}\right)\sum_{s=\pm1} E\{|\theta^D_{f-sf_{in}}|^2\}\right)} \tag{64}$$

In (64) we can directly substitute (61) which has the infinite sum over k already "built in."

Plots of two typical spectra based on the above equations are shown below. In blue, relative to the left Y-axis, the vertical bar DAC spectrum (64) and its dashed sinc-squared envelope are plotted, in units of A^2. The phase noise component is present but not visible on this scale, and the first term in (64) leads to clear non-zero values only at the ideal DAC tones. Therefore we plot in red, relative to the right Y-axis, on a 10^4 x smaller scale, the line-connected spectrum due to the second part in (64) *only* along with the sine-squared envelope for its maxima (as f_{in} changes), again in units of A^2. The top and bottom plot use input frequency bins of 3 and 13, respectively. Bin 16 corresponds to Nyquist, since 32 bins cover an f_{clk} interval. It is evident that the phase noise contribution is much worse for the second case while the main signal tones themselves are reduced, leading to a severe S/N decrease near Nyquist frequencies, which is quantified in (24) further below.

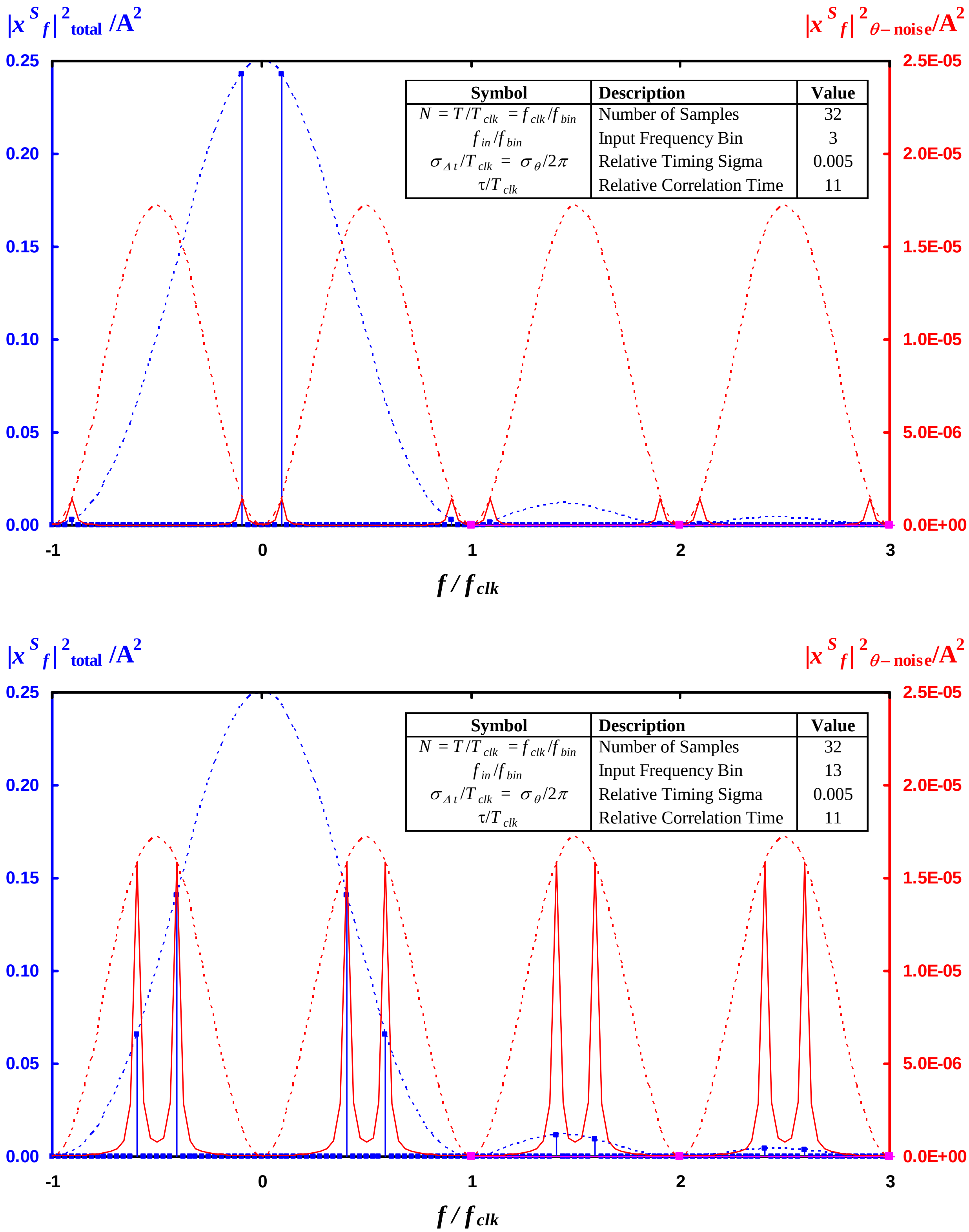

Figure 2. DAC output spectrum (blue, left Y-axis) and phase noise contribution only (red, right Y-axis, scale 104x smaller). Input frequency bins are 3 (top) and 13 (bottom) out of 32 total.

Total Power

The phase noise part in (54) or (64) repeats indefinitely in frequency with period f_{clk}, so would lead to infinite signal-plus-noise power in (55) if it were not for the small jitter assumption (31), which invalidates (54) for large absolute frequencies, $| f |$. In practice, finite output bandwidth more strongly limits the noise power contribution [16].

For independent identically distributed (*i.i.d.*) transition instant errors (white phase noise), the summand in the phase noise term in (64) becomes σ_θ^2/N in each frequency bin, giving a phase noise induced power

$$P_{\text{noise},\, f_{bin}\text{ interval}} = \frac{A^2}{2}\frac{1}{\pi^2}\sin^2\left(\frac{\pi f_{in}}{f_{clk}}\right)\frac{\sigma_\theta^2}{N} \tag{65}$$

and per Nyquist zone (f_{clk} period)

$$P_{\text{noise},\, f_{clk}\text{ interval}} = \frac{A^2}{2}\frac{1}{\pi^2}\sin^2\left(\frac{\pi f_{in}}{f_{clk}}\right)\sigma_\theta^2 \tag{66}$$

When the DAC is followed by a perfect $[-f_{clk}/2,\, f_{clk}/2]$ low-pass reconstruction filter, (66) gives *all* the DAC output noise power due to non-uniform sampling. If the output power were still $A^2/2$, which is indeed obtained analytically by summing the ideal lhs of (54) over all f, as in (55), see Appendix A, (71), the signal-to-noise ratio would correspond to the last equation in [17]. But the main signal (no noise) output power is reduced from $A^2/2$, as may be seen by simply summing the ideal first part of (54) from $-f_{clk}/2$ to $f_{clk}/2$ and assuming f_{in} is also in the first Nyquist zone:

$$P_{out,\text{1st Nyquist}} = \frac{A^2}{2}\text{sinc}^2\left(\frac{\pi f_{in}}{f_{clk}}\right) \tag{67}$$

This, of course, is simply the power in the two main DAC output tones. The corresponding signal-to-noise ratio is:

$$S/N = \left(\sigma_\theta \frac{f_{in}}{f_{clk}}\right)^{-2}, \tag{68}$$

which is Equation (7a) in [17]. In dB (69) reads

$$S/N\ (dB) = 20\log\left(\frac{f_{clk}}{\sigma_\theta f_{in}}\right) \tag{69}$$

For f_{in} approaching the Nyquist rate, $f_{clk}/2$, the worst S/N results: $20\log(2/\sigma_\theta)$. Related equations may be derived for output filters with different hard limits.

Error Power and Time Domain Analysis

The small jitter assumption (31) may actually be circumvented for *i.i.d.* (and still normally distributed) timing errors, equivalent to white phase noise. In that case an equation valid for all f can be derived for the spectrum that after infinite summation as in the rhs of (55) of the squared error relative to the ideal spectrum, gives a *finite* total "error power" equal to:

$$P_{\text{error}} = \sum_{f:f_{bin}=-\infty}^{\infty} E\{|\,x_f^F - x_f^F(\text{ideal})\,|^2\} = \sqrt{2\pi}\,\frac{A^2}{\pi^2}\sin^2\!\left(\frac{\pi f_{in}}{f_{clk}}\right)\sigma_\theta \tag{70}$$

Result (70) is identical to the result of a time domain approach [18] with $E\{|\theta|\} = (2/\pi)^{½}\sigma_\theta$, valid for normally distributed phase noise θ. This shows that the formalism is consistent and does not lead to infinite power even before taking finite DAC output bandwidth into account.

Both the quadratic dependence on amplitude, A, and the linear dependence on phase noise standard deviation, σ_θ, in (70) are intuitively correct for a zero-rise-time DAC. For small f_{in} / f_{clk}, the signal step sizes are proportional to f_{in}, so $f_{in}^{\ 2}$ makes sense too. For higher f_{clk} the steps become smaller (~ $f_{clk}^{\ -2}$) while the number of steps goes up (~ f_{clk}), and for a fixed σ_θ (parameter in (65)), it means the timing errors decrease (~ $f_{clk}^{\ -1}$), giving us a final $f_{clk}^{\ -2}$ dependence.

Note that (70) contains σ_θ linearly, while (65) depends on σ_θ quadratically. Dividing (70) by (65) gives the effective number of Nyquist zones counted in (70) as $2\pi^{½}/\sigma_\theta$, so very large in general, since normally $\sigma_\theta << 1$ [19]. This shows that finite output bandwidth will normally ensure that we are in the “quadratic jitter regime” of (65), and (54) or (64) may safely be used, possibly multiplied by the output filter transfer function.

Appendix A. Useful Sum Formulas

From 1.442(4) in [20] we obtain

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\alpha+k)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)} \tag{71}$$

This sum is referred to below (66).

Another useful sum is

$$\sum_{k=-\infty}^{\infty} \frac{1}{\alpha+k} = \frac{1}{\alpha} + \sum_{k=1}^{\infty} \left[\frac{1}{\alpha+k} + \frac{1}{\alpha-k}\right] = \frac{1}{\alpha} - \sum_{k=1}^{\infty} \frac{2\alpha}{k^2-\alpha^2} \tag{72}$$

With 1.445.6 of [20] with $x = 0$, $m = 0$, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{1}{\alpha+k} = \frac{1}{\alpha} - 2\alpha\left[\frac{1}{2\alpha^2} - \frac{\pi}{2}\frac{\cos(\pi\alpha)}{\alpha\sin(\pi\alpha)}\right] = \frac{\pi}{\tan(\pi\alpha)} \tag{73}$$

Identifying α as ν, the last equality in (33) is evident.

If we chose $x'(t)$ with plateaus x_n to be inside $[t_n - T_{clk}/2,\ t_{n+1} - T_{clk}/2\ [$, rather than our choice $[t_n,\ t_{n+1}\ [$ for x(t) in (25), one finds instead of (32)

$$x'^S_f = \operatorname{sinc}(\pi\nu)x^D_{in,f} + \frac{1}{N}\sum_{n=0}^{\infty} (x_n - x_{n-1})\, 1^{-\nu(n-1/2)} r_n \tag{74}$$

And (33) changes to

$$\begin{aligned} x'^D_{0,f} &\overset{(13)}{=} \sum_{k=-\infty}^{\infty} x'^S_{0,f+kf_{clk}} \overset{(62)}{=} x'^D_{in,f} \sum_{k=-\infty}^{\infty} \operatorname{sinc}(\pi(\nu+k)) = \\ &= 1^{-\nu/2} x'^D_{in,f} \frac{\sin(\pi\nu)}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\nu+k} \overset{(64)}{=} x'^D_{in,f} \end{aligned} \tag{75}$$

where instead of (73) we use

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\alpha+k} = \frac{\pi}{\sin(\pi\alpha)} \tag{76}$$

which is derived in a similar way, using 1.445.8 of [20].

Equation (75) shows that x'_n as derived from its SFT is equal to the input $x'_{in,n}$ if the continuous time $x'(t)$ is picked so as not to change value at t_n.

Appendix B. Expectation Value Theorem for Gaussian Variables

Theorem: If x and y are normally distributed with zero mean, variance $E\{x^2\} = E\{y^2\} = \sigma^2$, and covariance $E\{xy\} = R$, then for any real numbers p, q

$$E\{e^{i(px+qy)}\} = e^{-\frac{p^2+q^2}{2}\sigma^2 - pqR} \tag{77}$$

Proof: For a vector $\vec{x}$ of N normally distributed (or Gaussian) variables, the joint probability density function by definition has the form

$$p(\vec{x}) = \frac{1}{(2\pi)^{N/2}(\det R)^2} e^{-\frac{1}{2}(\vec{x}-E\{\vec{x}\})^T R^{-1}(\vec{x}-E\{\vec{x}\})} \tag{78}$$

for some N x N matrix R and N-vector $\vec{m}$ [1]. From (78) one can quickly prove

$$\vec{m} = E\{\vec{x}\} \qquad \text{“Mean Vector”} \tag{79}$$

and

$$R_{ij} = E\{(x_i - E\{x_i\})(x_j - E\{x_j\})\} \tag{80}$$

or equivalently

$$R = E\{(\vec{x} - E\{\vec{x}\})(\vec{x} - E\{\vec{x}\})^T\} \qquad \text{“Covariance Matrix”} \tag{81}$$

So in our case of x and y, the joint probability density function is:

$$p(x,y) = \frac{1}{2\pi\sqrt{\sigma^4 - R^2}} e^{-\frac{\sigma^2(x^2+y^2)-2Rxy}{2(\sigma^4-R^2)}} \tag{82}$$

Calculation of the lhs of (77) with

$$E\{e^{i(px+qy)}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(px+qy)} p(x,y)dxdy = \frac{1}{2\pi\sqrt{\sigma^4 - R^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(px+qy)-\frac{\sigma^2(x^2+y^2)-2Rxy}{2(\sigma^4-R^2)}} dxdy \tag{83}$$

leads to the rhs of the same. **QED**

Choosing $p = \nu$ and $q = -\nu$, (77) results in (37) of the main text.

Appendix C. Coherent vs. Incoherent Power Addition

In going from (39) to (43) one encounters the square of the DFT of $x_{in,n}$, or

$$|x_{in,f}^D|^2 = \left(\frac{A}{2}\right)^2 \left(\sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} \right)^2 = \left(\frac{A}{2}\right)^2 \sum_{\substack{k,k'=-\infty \\ s,s'=\pm 1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} \delta_{f,s'f_{in}+k'f_{clk}} \quad (84)$$

In the rhs of (84) we note that for f_{in} an integer multiple of $f_{clk}/2$ there are two ways in each of the δ functions to equal 1. The result may be written as:

$$|x_{in,f}^D|^2 = \left(\frac{A}{2}\right)^2 \times \begin{cases} \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} & \text{if } f_{in} \neq \text{integer} \times f_{clk}/2 \\ 4\sum_{k=-\infty}^{\infty} \delta_{f,kf_{clk}} & \text{if } f_{in} = \text{integer} \times f_{clk} \\ 4\sum_{k=-\infty}^{\infty} \delta_{f,(k+1/2)f_{clk}} & \text{if } f_{in} = (\text{integer}+1/2) \times f_{clk} \end{cases} \quad (85)$$

Using the auxiliary quantity

$$S_{in} = \begin{cases} 1 \text{ if } f_{in} \neq \text{integer} \times f_{clk}/2 \\ 2 \text{ if } f_{in} = \text{integer} \times f_{clk}/2 \end{cases} \quad (86)$$

(85) may be written as the single identity

$$|x_{in,f}^D|^2 = S_{in} \left(\frac{A}{2}\right)^2 \sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f,sf_{in}+kf_{clk}} \quad (87)$$

Using (87), (43) in the main text follows easily from (39).

The result (86), (87) depends on the phase of the wave $x_{in,n}$. The cosine wave (41) represents zero phase ϕ in the more general wave

$$x_{in,n} = \frac{A}{2}\sum_{s=\pm 1} 1^{s(\nu_{in}+\frac{\phi}{2\pi})n} = \frac{A}{2}\sum_{s=\pm 1} e^{is(2\pi\nu_{in}+\phi)n} \quad (88)$$

Still (87) applies but with this S_{in}:

$$S_{in} = 1 + \cos(2\phi) \sum_{k=-\infty}^{\infty} \delta_{f_{in},kf_{clk}/2} = 1 + (2\cos^2(\phi) - 1) \sum_{k=-\infty}^{\infty} \delta_{f_{in},kf_{clk}/2} \quad (89)$$

or

$$S_{in} = \begin{cases} 1 \text{ if } f_{in} \neq \text{integer} \times (N/2) \\ 2\cos^2(\phi) \text{ if } f_{in} = \text{integer} \times (N/2) \end{cases} \quad (90)$$

which generalizes (86).

According to Wiener-Khinchin (18), (87) is the DFT of the $x_{in,n}$ correlation function, $c_{x,n}$, which indeed works out to be

$$c_{x,n} = \frac{1}{N}\sum_{m=0}^{N-1} x_m^* x_{m+n} = S_{in}\frac{A^2}{2}\cos(2\pi\nu_{in} n) \tag{91}$$

Anomalous spectral power (87) only occurs for half-integer multiples of the clock frequency, and not for quarter-integer multiples say, because correlation functions involve a product of two, not more, instances of $x_{in,n}$, see (91). We can think of the anomalous cases as resulting from a "coherent" addition of N identical products in the autocorrelation function $c_{x,n}$. We note that averaging (90) over phase ϕ always gives 1.

Notes and References

[1] A. J. Viterbi, Principles of Coherent Communication, McGraw-Hill, 1966.

[2] A complimentary case of nonuniformly sampled input and no jitter on the DAC output clock is investigated in Y. C. Jenq, Digital-to-analog (D/A) converters with nonuniformly sampled signals, IEEE Trans. Instrumentation Meas., vol. 45, pp. 56-59, Feb. 1996.

[3] The notation "$:f_{bin}$" under the summation symbol indicates that frequency f steps by f_{bin}.

[4] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes in C, Cambridge University Press, Second Edition, 1992, Chapter 12. Like these authors, we absorb 2π in the exponent, which leads to more symmetrical equations with simpler pre-factors. Note however that their Fourier transform pair convention makes $H(f)$ in (12.0.1) (or H_n in (12.1.9)) the "weight" of frequency component $-f$ ($-n$) rather than f (n) as in our convention.

[5] The series x_n repeats with period N, but this does not matter to us.

[6] The function $x(t)$ repeats with period T, but this does not matter to us.

[7] A very accessible discussion of some of these can be found in [4]

[8] The more general case involving a product of two series or functions is also called Parseval's theorem. That one in turn allows quick derivation of (16) and (17) by choosing the second function a n (t) shifted version of x. Other names associated with this theorem are Plancherel and Rayleigh.

[9] Our z are ultimately real. But for generality, we give the equations for complex z.

[10] J. Hinrichs and G. Miao, Jitter Error Spectrum for NRZ D/A Converters, Proc. IEEE Int. Symp. Circ. Systems (ISCAS), May 18-21, 2008, pp. 2410-2413.

[11] Y. C. Yenq, Fourier Spectrum of D/A Outputs with Non-uniformly Sampled Data and Time-Varying Clocks, Proc. 4th Conference on Advanced A/D and D/A Conversion Techniques and their Applications and 7th Workshop on ADC Modeling and Testing, Prague, Czech Republic, June 26-28, 2002. http://www.imeko.org/publications/tc4-2002/IMEKO-TC4-2002-018.pdf. See section 4.

[12] If x is a voltage over a resistor R, then x^2/R is the instantaneous power dissipation in R.

[13] If one uses continuous integration rather than summation in the frequency domain, the power spectrum is defined as $\sum_{f':f_{bin}=-\infty}^{\infty} |x_f^S|^2\,\delta(f-f')$, see Sec. 6-2, p. 90 of W. B. Davenport and W. L. Root, An Introduction to the Theory of Random Signals and Noise, IEEE Press, 1987 reprint.

[14] B. Brannon, Sampled Systems and the Effects of Clock Phase Noise and Jitter, Analog Devices Application Note AN-756.

[15] If the clock passes through an electronic circuit with a dominant pole, τ will be close to that pole's time.

[16] There are at least five reasons to design (including filter) for the lowest output bandwidth possible: (1) The usual reconstruction filtering, i.e. suppress the DAC image tones beyond Nyquist, (2) Reduce the total phase noise induced power which is under discussion here, (3) Reduce additive noise by DAC output circuitry, (4) Reduce interference to other circuitry, including through power supplies, and (5) Reduce reflections from imperfect output terminations. Additionally, it may allow power reduction, and if transmitting off-chip over great distance, may decrease sensitivity to dispersion (frequency dependent delay).

[17] P. Smith, Little Known Characteristics of Phase Noise, Analog Devices Application Note AN-741.

[18] N. Kurosawa, H. Kobayashi, H. Kogure, T. Komuro, H. Sakayori, Sampling clock jitter effects in digital-to-analog converters, Measurement 31, pp. 187-199, 2002. See their Proposition on p. 189. On that page it is printed with an error (extra "n"). It is correctly printed at the end of their appendix A, p. 198. In their equation, substitute $E\{|\varepsilon|\} = E\{|\theta/2\pi|\} = (2/\pi)^{½}\sigma_\theta / 2\pi = 2^{-1/2}\pi^{-3/2}\sigma_\theta$, valid for normally distributed phase noise.

[19] If σ_θ is not much smaller than one, very bad S/N results according to (24) and this is not the case of practical interest.

[20] I. S. Gradsteyn and I. M. Ryzhik, A. Jeffrey Ed., Table of Integrals, Series, and Products. Academic Press, 5th ed., San Diego, 1994.