Finite Resolution DAC Spectrum

Paul van der Wagt, May 2014

The output signal spectrum of a finite resolution digital-to-analog converter (DAC) with a sinusoidal input is expressed as a sum over all DAC output steps. The formulation includes arbitrary static differential non-linearity (DNL). The exact result allows fast numerical evaluation, bypassing explicit time domain computation followed by Fourier transformation. Two appendices describe a key summation formula and the exact output spectrum of a linear finite resolution analog-to-digital converter (ADC).

Notation, Assumptions, and Tools

June 2010, Carlsbad, California

1^z A shorthand notation for any complex number z:

$$1^z = e^{2\pi iz} \tag{1}$$

Δ_{in} The DAC digital input code step. This is dimensionless and normally equals 1. It is kept here for optional input scaling. For application of some of our results to ADCs, Δ_{in} would have the dimension of the input signal and if the ADC is non-linear, $n\Delta_{in}$ can be replaced by a function of threshold index $n = -q', -q'+1, \ldots, -1, 0, 1, \ldots, q'-1, q'$.

$\Delta_{out,n}$ The analog DAC output steps as a function of threshold index $n = -q', -q'+1, \ldots, -1, 0, 1, \ldots, q'-1, q'$. In the case of an ADC with integer output codes, for all n, $\Delta_{out,n} = 1$.

q' $Q/2 - 1$. Summations in many of our equations run from $-q'$ to q'.

Q Number of DAC output levels, assumed even.

FSR_{in} Full scale input range; equals $Q\,\Delta_{in}$.

f_{clk} Nominal clock frequency. Rate at which input to output data conversion happens.

T_{clk} Clock period; equals $1 / f_{clk}$.

N An integer parameter setting the total measurement time, T. Also equal to the number of bins in our DFT.

T NT_{clk}, measurement time.

t Time. The DAC input reference signal, the DAC output signal, and the ADC input signal are functions of this continuous parameter.

f_{bin} Bin size in frequency space; equal to $1/T = f_{clk}/N$.

f Frequency. Only values $f = p\, f_{bin}$, p integer, are considered here.

f_{in} Frequency of ideal input signal. It is assumed that $f_{in} = p_{in}\, f_{bin}$, p_{in} integer. This means that $p_{in}\,(1/f_{in}) = 1/f_{bin} = T$.

x_0, x_0' Offset of signal $x_{in}(t)$. Only $x_0' = x_0/FSR_{in}$ enters final equations.

A, A' Amplitude of signal $x_{in}(t)$. Only $A' = A/FSR_{in}$ enters final equations, and a full scale sine wave corresponds to $A' = 0.5$.

ϕ_0 Phase offset of signal $x_{in}(t)$ relative to a cosine wave.

$x_{in}(t)$ DAC input reference signal and ADC input signal as a continuous function of time, t:

$$x_{in}(t) = x_0 + A\cos(\phi_0 + 2\pi f_{in} t) \tag{2}$$

For the DAC, $x_{in}(t)$ is first ideally vertically quantized into a digital sine wave. See later section on DAC definition.

$x(t)$ Unclocked DAC output signal as a continuous function of time, t.

$x^E(t)$ DAC output signal as a continuous function of time, t. The "E" stands for "extended."

$x_{in,n}$ Value of sinusoidal wave sampled uniformly with period T_{clk}:

$$x_{in,n} = x_{in}(nT_{clk}) = A\cos(2\pi f_{in} nT_{clk}), \; n \text{ integer} \tag{3}$$

x_n The x_n, $n = 0, 1, \ldots, N$, are the DAC output plateaus during $[(n - ½)T_{clk}, (n + ½)T_{clk}[$. The The DAC has finite resolution and may be non-linear, but the output transition times are ideal.

x_f^D Discrete Fourier transform (DFT) of a discrete function x_n, a function of discrete frequency, f. The DFT pair is [1][2]:

$$x_n = \sum_{f:f_{bin}=0}^{f_{clk}-f_{bin}} x_f^D 1^{fnT_{clk}}, \; n = 0, 1, \ldots, N-1 \tag{4}$$

$$x_f^D = \frac{1}{N}\sum_{n=0}^{N-1} x_n 1^{-fnT_{clk}}, \; f = p\, f_{bin}, p \text{ any integer} \tag{5}$$

Note that we allow f outside the interval $[0, f_{clk}]$, so the DFT repeats with period f_{clk} [3]. For a sinusoidal wave (3):

$$x_{in,f}^D = \frac{A}{2}\sum_{\substack{k=-\infty \\ s=\pm 1}}^{\infty} \delta_{f, sf_{in}+kf_{clk}} \tag{6}$$

x_f^S Series Fourier transform (SFT) of a continuous function $x(t)$, a function of discrete frequency, f. The SFT pair is [2]

$$x(t) = \sum_{f:f_{bin}=-\infty}^{\infty} x_f^S 1^{ft}, \; t \in [0, T[\tag{7}$$

$$x_f^S = \frac{1}{T}\int_0^T x(t) 1^{-ft} dt, \; f = p\, f_{bin}, p \text{ any integer} \tag{8}$$

The SFT does not repeat with period f_{clk}: Components with arbitrarily large f in (7) add to the description of the continuous function $x(t)$ [4]. For a sinusoidal wave (3):

$$x_{in,f}^S = \frac{A}{2}\sum_{s=\pm 1} \delta_{f, sf_{in}} \tag{9}$$

From the definitions above one can prove the all-important aliasing relationship between DFT and SFT [5]:

$$x_f^D = \sum_{k=-\infty}^{\infty} x_{f+kf_{clk}}^S \qquad \text{"Aliasing Theorem"} \tag{10}$$

A trivial example of (10) is provided by (6) and (9). Each side of (10) depends only on the values $x(nT_{clk})$, even though individual x^S do depend on the full $x(t)$. The summation effectively "focuses" all the weight on a discrete subset of times. A corollary is that the x used on either side does not need to be the same function: Only the values at $t = nT_{clk}$ need agree. We make use of this fact later.

DAC Definition

July 2013, Dominican Republic, Punta Cana, Playa Arena Blanca

We consider a mid-riser digital-to-analog converter (DAC) with an even number, Q, of quantized output levels. By definition of "mid-riser," the input transition thresholds are at $n\Delta_{in}$, $n = -q', -q'+1, \ldots, -1, 0, 1, \ldots, q'-1, q'$, where $q' = Q/2 - 1$. There are an odd number, $Q - 1$, of thresholds $n\Delta_{in}$. The un-clocked version of this DAC has the transfer function

$$x = o + \sum_{n=-q'}^{q'} \Delta_{out,n}\,\theta\!\left(\frac{x_{in}}{\Delta_{in}} - n\right) \tag{11}$$

where θ is the Heaviside step function (0, 1 for negative, non-negative argument, respectively) and o is the output for inputs below the first threshold. This "first level" is also the lowest output if the DAC steps $\Delta_{out,n}$ are positive. The $Q - 1$ arbitrary output steps $\Delta_{out,n}$ allow for static DAC non-linearity. If the DAC is perfectly linear, all the $\Delta_{out,n}$ are equal to Δ_{out} and $o = -q'\Delta_{out}$. Figure 1 shows an example non-linear DAC transfer curve.

What about cosine waves going beyond the effective DAC input range? Such 'overload' conditions are known to lead to very large distortion levels. The surprisingly simple answer is that this is captured by the two indefinitely extending first and last levels (see also Figure 1) and the fact that our subsequent derivations based on (11) are exact, not approximate.

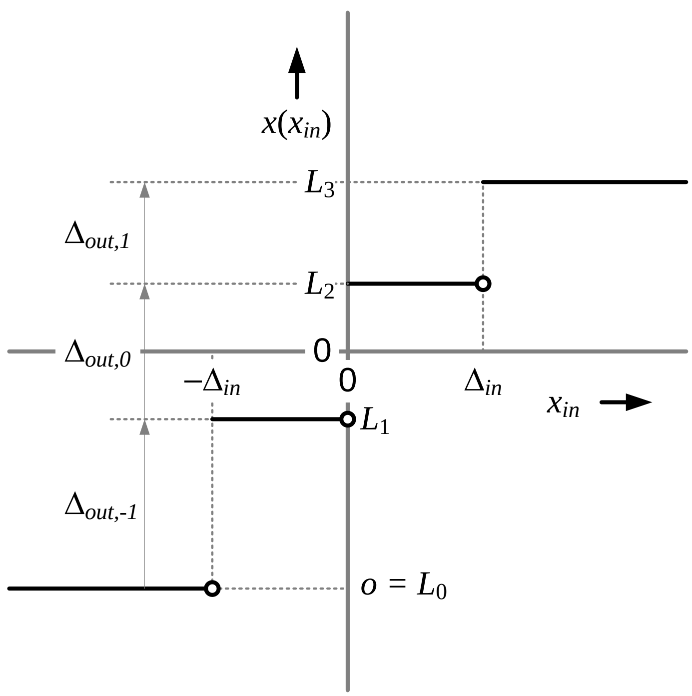

Figure 1. Example mid-riser DAC with Q = 4 levels.

For the general case, the DAC output levels L_l, $l = 0, 1, 2, \ldots, Q - 1$, are:

$$L_l = o + \sum_{n=-q'}^{-q'+l-1} \Delta_{out,n} \; , \; l = 0, 1, 2, \ldots, Q-1 \; (= 2q'+1) \tag{12}$$

If a nominal output step Δ_{out} has been defined from the slope of a line through the end points (other conventions are possible, but typically are not quite as simple) then

$$\Delta_{out,n} = \Delta_{out}(1 + DNL_n),\; n = -q', -q'+1, \ldots, -1, 0, 1, \ldots, q'-1, q' \tag{13}$$

The input scale step Δ_{in} may simply be taken as 1, but is carried here for clarity. In later results we should always find x_{in}-unit quantities normalized to Δ_{in}.

The clocked DAC output levels, x_n, follow trivially from (11)

$$x_n = x(nT_{clk}) = o + \sum_{j=-q'}^{q'} \Delta_{out,n} \theta\left(\frac{x_{in}(nT_{clk})}{\Delta_{in}} - j \right) \tag{14}$$

The complete time-domain clocked DAC output requires a "block extension" x^E from (14). We first define x^E to equal x at times nT_{clk}:

$$x_n^E = x^E(nT_{clk}) = x(nT_{clk}) = x_n \tag{15}$$

The continuous time extension then is

$$x^E(t) = \sum_{n=0}^{N-1} B((n - \tfrac{1}{2})T_{clk}, (n + \tfrac{1}{2})T_{clk}; t) x_n^E \tag{16}$$

where the "block" function B is given by

$$B(a, b; t) = \begin{cases} 0 \text{ if } \; t \notin [a, b[\\ 1 \text{ if } \; t \in [a, b[\end{cases} \text{, assumes } a < b \tag{17}$$

see Figure 2. Figure 3 illustrates the functions we just introduced.

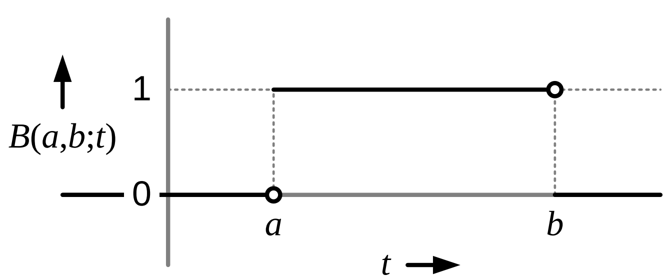

Figure 2. Function $B(a, b; t)$ of t for given $a < b$.

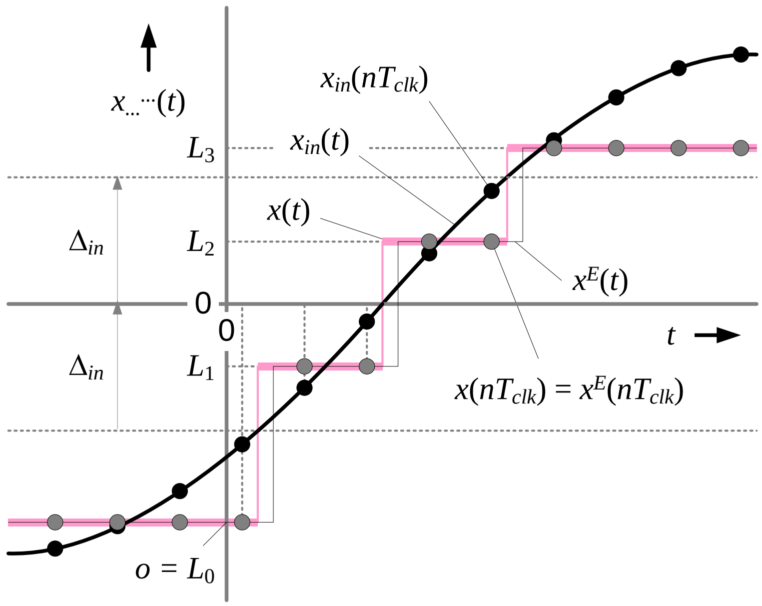

Figure 3. Relationship of x_{in}, x, x^E for a four-level DAC example. The vertical scale meaning depends on the function plotted. In this example the input step Δ_{in} approximately equals the average of the output steps $\Delta_{out,n}$ (n = -1, 0, 1). Note that $\Delta_{out,n} = L_{n+q'+1} - L_{n+q}$, i.e. $\Delta_{out,-1} = L_1 - L_0$, etc.

DAC Output Spectrum Derivation

1996 - 1997, Dallas, TX, Texas Instruments and
July 2013, Dominican Republic, Punta Cana, Playa Arena Blanca

The derivation of the DAC output spectrum for a given $x_{in}(t)$ proceeds as follows:

1. Determine the SFT $x^S{}_f$ of the un-clocked DAC output $x(t)$ in (11)
2. Use the aliasing theorem (10) to obtain the DFT $x^D{}_f$, which only deals with $x(nT_{clk})$
3. Obtain DFT $x^{ED}{}_f$ from $x^{ED}{}_f = x^D{}_f$, since $x^E(t) = x(t)$ at $t = nT_{clk}$
4. Derive the SFT of the clocked DAC output (16) $x^{ES}{}_f$ from $x^{ED}{}_f$

This sequence involves manageable pieces and shows interrelationships with well known results. The 3rd and 4th items above are relatively quick, while the 1st and 2nd contain the bulk of the derivation. The steps show a high-level "back-and-forth" action between SFTs and DFTs that tends to complicate the overall derivation if left implicit. In fact, the author obtained key results in this paper 18 years ago using rather *ad-hoc* methods and converged only recently on this more streamlined presentation.

Step 1: SFT of Un-clocked DAC Output with Sinusoidal Input

From (2) and (11) the un-clocked (continuous time) DAC output SFT is:

$$
\begin{aligned}
x_f^S &= \frac{1}{T}\int_0^T x(t)1^{-ft}\,dt = \\
&= \frac{1}{T}\int_0^T \left[o + \sum_{n=-q'}^{q'} \Delta_{out,n}\,\theta\!\left(\frac{x_0 + A\cos(\phi_0 + 2\pi f_{in}t)}{\Delta_{in}} - n\right)\right]1^{-ft}\,dt = \\
&= o\delta_{f,0} + \frac{1}{T}\sum_{n=-q'}^{q'} \Delta_{out,n} e^{if\phi_0/f_{in}} \int_0^T \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(2\pi f_{in}t)\right)1^{-ft}\,dt
\end{aligned}
\tag{18}
$$

The integrals will be nonzero only if $f = hf_{in}$, h integer. So:

$$
x_f^S = o\delta_{f,0} + \frac{e^{ih\phi_0}}{T}\sum_{n=-q'}^{q'} \Delta_{out,n}\int_0^T \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(2\pi f_{in}t)\right)1^{-hf_{in}t}\,dt
\tag{19}
$$

Now evaluate the integrals:

$$
\begin{aligned}
&\frac{1}{T}\int_0^T \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(2\pi f_{in}t)\right)1^{-hf_{in}t}\,dt = \\
&= \frac{1}{2\pi}\int_0^{2\pi} \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(p_{in}\phi)\right)e^{-hp_{in}\phi}\,d\phi = \\
&= \frac{p_{in}}{2\pi}\int_0^{2\pi/p_{in}} \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(p_{in}\phi)\right)e^{-hp_{in}\phi}\,d\phi = \\
&= \frac{1}{2\pi}\int_0^{2\pi} \theta\!\left(\frac{x_0 - n\Delta_{in}}{A} - \cos(\phi')\right)e^{-h\phi'}\,d\phi' = \\
&= \frac{1}{\pi h}\sin\!\left(h\cos^{-1}\!\left(\frac{n\Delta_{in} - x_0}{A}\right)\right)
\end{aligned}
\tag{20}
$$

where $\cos^{-1}$ is an extended version of the arccos function:

$$
\cos^{-1}(x) = \begin{cases} \pi\ ,\ x \le -1 \\ \arccos(x)\ ,\ x \in [-1,1] \\ 0\ ,\ x \ge 1 \end{cases}
\tag{21}
$$

This function is shown in Figure 4. Without this extension (20) fails for cases where the argument of the Heaviside step function θ does not change sign during the integration. This happens for DAC levels that are either completely below or above the input cosine wave. In other words, the extension is key to handling the input under-load case and over-load case correctly.

Substitute (20) into (19):

$$
x_{hf_{in}}^S = o\delta_{h,0} + \frac{e^{ih\phi_0}}{\pi h}\sum_{n=-q'}^{q'} \Delta_{out,n}\sin(h\alpha_n)
\tag{22}
$$

with x^S_f zero for f not equal to hf_{in} and where

$$\alpha_n = \cos^{-1}\left(\frac{n\Delta_{in} - x_0}{A}\right) = \cos^{-1}\left(\frac{n/Q - x_0'}{A'}\right) \tag{23}$$

Where x_0' and A' are x_0 and A scaled relative to $FSR_{in} = Q\,\Delta_{in}$. The rhs. of (22) depends indirectly on f_{in}, since h signifies the harmonics of f_{in}.

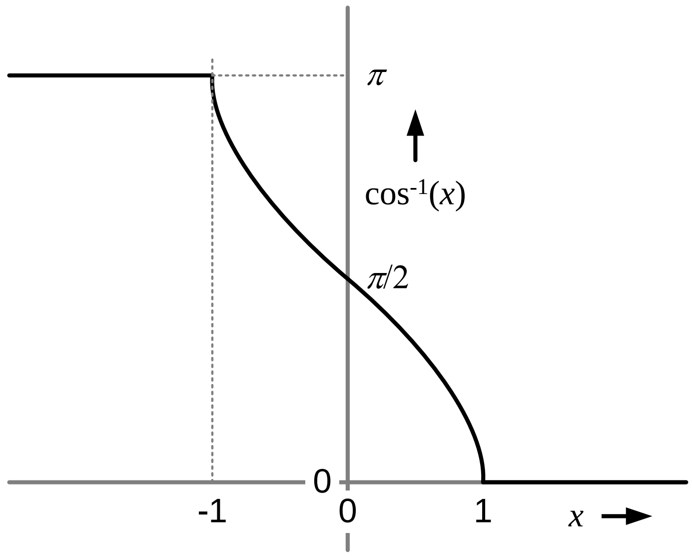

Figure 4. Extended inverse cosine function.

Step 2: DFT of Un-clocked DAC Output with Sinusoidal Input

We use the aliasing theorem to obtain the DFT from the SFT.

$$\begin{aligned} x_f^D &= \sum_{\nu=-\infty}^{\infty} x_{f+\nu f_{clk}}^S = \\ &= \sum_{\nu=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \delta_{f+\nu f_{clk}, hf_{in}} x_{f+\nu f_{clk}}^S = \\ &= \sum_{h=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} \delta_{f+\nu f_{clk}, hf_{in}} x_{hf_{in}}^S = \\ &= \sum_{h=0}^{N-1} \delta_{(f, hf_{in}) \bmod f_{clk}} \sum_{m=-\infty}^{\infty} x_{(h+mN) f_{in}}^S \end{aligned} \tag{24}$$

If p_{in} and N are relatively prime, i.e. GCD(p_{in},N) = 1 (and we would call this N for $p_{in} = 0$: We are in modulo-N land here), $hf_{in} \bmod f_{clk} = (hp_{in} \bmod N)f_{bin}$ traverses all frequency bins 0, f_{bin}, $2f_{bin}$, …, $(N-1)f_{bin}$, but in a different order (unless $p_{in} = 1$). In this case there is a unique h for each $f = 0$, f_{bin}, $2f_{bin}$, …, $(N-1)f_{bin}$, and (24) may be simplified to:

$$x_{hf_{in} \bmod f_{clk}}^D = \sum_{m=-\infty}^{\infty} x_{(h+mN) f_{in}}^S \tag{25}$$

But for general input frequencies we should keep (24). If GCD(p_{in},N) > 1, $x^D{}_f$ will be zero for some f while for other f we get multiple contributions in (24). So the front h-sum in (24) acts as a redistribution of sorts of the back m-sums into $x^D{}_f$.

Turning our attention to the m-sums:

$$\begin{aligned}\sum_{m=-\infty}^{\infty} x^S_{(h+mN)f_{in}} &= \sum_{m=-\infty}^{\infty}\left[o\delta_{h+mN,0} + \frac{e^{i(h+mN)\phi_0}}{\pi(h+mN)}\sum_{n=-q'}^{q'}\Delta_{out,n}\sin((h+mN)\alpha_n)\right] = \\ &= o\delta_{h,0} + \frac{1}{\pi}\sum_{n=-q'}^{q'}\Delta_{out,n}\sum_{m=-\infty}^{\infty}\frac{e^{i(h+mN)\phi_0}\sin((h+mN)\alpha_n)}{h+mN} = \\ &= o\delta_{h,0} + \frac{1}{2\pi i}\sum_{n=-q'}^{q'}\Delta_{out,n}\sum_{m=-\infty}^{\infty}\left[\frac{e^{i(h+mN)(\phi_0+\alpha_n)}}{h+mN} - \frac{e^{i(h+mN)(\phi_0-\alpha_n)}}{h+mN}\right] = \\ &= o\delta_{h,0} + \frac{1}{2\pi i}\sum_{n=-q'}^{q'}\Delta_{out,n}(S(\phi_0+\alpha_n) - S(\phi_0-\alpha_n))\end{aligned} \tag{26}$$

where

$$S(\phi) = \sum_{m=-\infty}^{\infty}\frac{e^{i(h+mN)\phi}}{h+mN} = \frac{\frac{\pi}{N}}{\sin(\frac{\pi h}{N})}e^{\frac{2\pi ih}{N}MR\left(\frac{N\phi}{2\pi}\right)} \tag{27}$$

The last equality requires a non-trivial derivation presented in Appendix A. The function MR stands for "mid-riser" and is given by

$$MR(x) = E(x) + \tfrac{1}{2} = x \bmod 1 + \tfrac{1}{2} \tag{28}$$

where $E(x)$ is the Entier function, equivalent to integer round down, a.k.a. the "floor" function. Equation (27) collapses an infinite sum to a simple function and represents a massive breakthrough. Substitution of (27) into (26) gives

$$\begin{aligned}\sum_{m=-\infty}^{\infty} x^S_{(h+mN)f_{in}} &= o\delta_{h,0} + \frac{1}{2iN\sin(\frac{\pi h}{N})}\sum_{n=-q'}^{q'}\Delta_{out,n}\left[e^{\frac{2\pi ih}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - e^{\frac{2\pi ih}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)}\right] = \\ &= o\delta_{h,0} + \frac{1}{2iN\sin(\frac{\pi h}{N})}\sum_{n=-q'}^{q'}\Delta_{out,n}\left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)}\right]\end{aligned} \tag{29}$$

The rhs. of (29) depends only indirectly on f_{in}, since h signifies the harmonic number of f_{in}.

Combining (24) and (29) we have:

$$\begin{aligned}x^D_f &= \sum_{h=0}^{N-1}\delta_{(f,hf_{in})\bmod f_{clk}}\left[o\delta_{h,0} + \frac{1}{2iN\sin(\frac{\pi h}{N})}\sum_{n=-q'}^{q'}\Delta_{out,n}\left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)}\right]\right] = \\ &= o\delta_{f \bmod f_{clk},0} + \sum_{h=0}^{N-1}\frac{\delta_{(f,hf_{in})\bmod(f_{clk})}}{2iN\sin(\frac{\pi h}{N})}\sum_{n=-q'}^{q'}\Delta_{out,n}\left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)}\right]\end{aligned} \tag{30}$$

Appendix B discusses the application of this to a linear finite resolution ADC.

Step 3: DFT of Clocked DAC Output with Sinusoidal Input

From (15) we have

$$x_n^E = x^E(nT_{clk}) = x(nT_{clk}) = x_n \tag{31}$$

and hence

$$x_f^{ED} = x_f^D \tag{32}$$

With (30) this means

$$x_f^{ED} = o\delta_{f \bmod f_{clk},0} + \sum_{h=0}^{N-1} \frac{\delta_{(f,hf_{in}) \bmod (f_{clk})}}{2iN\sin(\frac{\pi h}{N})} \sum_{n=-q'}^{q'} \Delta_{out,n} \left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)} \right] \tag{33}$$

This right hand side is simply equal to that of (30).

Step 4: SFT of Clocked DAC Output with Sinusoidal Input

The SFT of B with respect to t is

$$[B(a,b;t)]_f^S = \frac{1}{T}\int_0^T B(a,b;t)1^{-ft}\,dt = \frac{1^{-fa} - 1^{-fb}}{2\pi ifT} \tag{34}$$

The SFT of the DAC output follows from (16) and (34) where we make use of $x_N = x_0$:

$$\begin{aligned} x_f^{ES} &= \frac{1}{T}\int_0^T \sum_{n=0}^{N} B((n-\tfrac{1}{2})T_{clk},(n+\tfrac{1}{2})T_{clk};t)x_n^E 1^{-ft}\,dt = \\ &= \frac{1}{2\pi ifT}\sum_{n=0}^{\infty}(1^{-f(n-\frac{1}{2})T_{clk}} - 1^{-f(n+\frac{1}{2})T_{clk}})x_n^E = \frac{1}{2\pi ifT_{clk}}(1^{\frac{1}{2}fT_{clk}} - 1^{-\frac{1}{2}fT_{clk}})x_f^{ED} = \\ &= \mathrm{sinc}(\pi fT_{clk})x_f^{ED}\ ,\ f = pf_{bin}, p \text{ any integer} \end{aligned} \tag{35}$$

As a check on both (10) and (35) we derive:

$$\begin{aligned} x_f^D &\overset{(26)}{=} x_f^{ED} \overset{(10)}{=} \sum_{k=-\infty}^{\infty} x_{f+kf_{clk}}^{ES} \overset{(29)}{=} x_f^D \sum_{k=-\infty}^{\infty} \mathrm{sinc}(\pi(fT_{clk}+k)) = \\ &= x_f^D \frac{\sin(\pi fT_{clk})}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{fT_{clk}+k} \overset{(xxx)}{=} x_f^D \end{aligned} \tag{36}$$

For the last equality, see (47) in Appendix A.

Combining (33) and (35) gives

$$x_f^{ES} = o\delta_{f \bmod f_{clk},0} + \mathrm{sinc}(\tfrac{\pi f}{f_{clk}}) \sum_{h=0}^{N-1} \frac{\delta_{(f,hf_{in}) \bmod (f_{clk})}}{2iN\sin(\frac{\pi h}{N})} \sum_{n=-q'}^{q'} \Delta_{out,n} \left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)} \right] \tag{37}$$

This is the complete expression for the non-ideal-DAC output spectrum for a sinusoidal input. The only difference on the right hand side relative to (30) is the sinc factor.

Numerical Examples

May 2014, Carlsbad, California

We conclude with results from example spreadsheet calculations to visualize specific cases. As mentioned before, our equations remove the need for a transient DAC or ADC simulation followed by Fourier transformation. Since (37) is so simply related to (30) through (35), we will focus on (30). In other words, we calculate the DAC spectrum before the sinc correction. For perfectly linear DAC, i.e. all Dout,n equal, this also is the spectrum of a perfectly linear ADC, see Appendix B.

Example 1 is an 8-bit perfectly linear DAC with a full scale sinusoidal output (really a full-scale input, but it leads to a full-scale output). For a 1-V output FSR scale, the energy in the main output sine wave is 0.124935, very close to the limiting value of 0.125 (1/8) for infinite resolution. The ENOB is 7.92, slightly below the nominal value 8. The non-equality is due to the fact that the ENOB definition used (SNR = 6.02 ENOB + 1.76) is only exact in the limit of infinite resolution. Minor changes in simulation parameters, for instance input signal phase, give ENOBs both slightly above and slightly below 8, see example 2.

Example 2 is the same DAC, but the input signal phase is 0.1 rad rather than 0 rad. Numbers change a little, and in particular the ENOB is now 8.10, so above 8.

Example 3 is the same DAC, but the amplitude is reduced by 16x (and input signal phase back to 0 rad). The ENOB drops to 3.90, in line with expectations, since this should be identical to a 4-bit DAC, see example 4.

Example 4 is a 4-bit perfectly linear DAC with a full scale sinusoidal output. The result is identical to example 3, except that all amplitudes are 16x smaller. Relative measures, including ENOB, are identical.

Example 5 is a 1-bit DAC, and here the ideal output spectrum (before sinc correction) is very well known. In fact, apart from a scale factor, it is given by (58) in Appendix B. We checked explicitly that the harmonic behavior follows this expected behavior. Note that in the 1-bit case the calculated ENOB is more inaccurate than usual, since the approximations involved in the equation are worse.

Example 6 shows the detrimental effect of 5% amplitude overload: The ENOB collapses from 7.92 to 5.33. Our equations correctly account for this effect. Due to the symmetrical nature of the error, only odd harmonics appear.

Example 7 shows a similar effect from a 6.25% (1/16) offset relative to the full scale range (FSR). Since the amplitude is already at full scale, we now have positive overload only. Due to this asymmetric nature both even and odd order harmonics appear, including at DC. The energy at DC is not simply the square of the offset due to the sine wave distortion at its peak.

Example 8 shows that such an offset has no effect on ENOB as long as we ensure that no overload occurs: The amplitude was reduced from 0.5 to 0.4375. Now the DC energy (0.003906) equals the square of the offset (0.0625).

Example 9 is the reference case for example 8. The ENOBs are equal since the offset was chosen as a multiple of the DAC level spacing. Otherwise, small ENOB fluctuations tend to result.

So far the examples have been for perfectly linear DACs. But our formalism handles cases with arbitrary DAC levels. We add a few cases where we define DNL with a uniform random distribution in a certain interval relative to one LSB. The effect of DNL enters into (30) through

$$\Delta_{out,n} = \Delta_{out,n}(1 + DNL_n) \tag{38}$$

The amplitude is kept below full scale to ensure that for various DNL range settings no overload occurs.

Examples 10, 11, 12, 13, and 14 show cases of DNL uniformly distributed between -A_{DNL} LSB and A_{DNL} LSB, for A_{DNL} is 0.125, 0.25, 0.5, 1, and 2, respectively. We can interpret example 9 as the case of $A_{DNL} = 0$. The ENOBs for the six cases in order of increasing DNL are 7.66, 7.55, 7.18, 6.47, 5.58, and 4.69. Note that we apply a “frozen” random DNL pattern for all cases and merely adjust its amplitude. For every doubling of DNL amplitude, we do not get a full ENOB bit loss, initially because the DNL is small compared to the inherent quantization step limiting ENOB to about 7.66 for the chosen amplitude, and later due to the fact that the main tone energy starts to increase as a result of the random DNL increase.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.5**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.124935	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.2499	V^2
Energy, Signal		Harmonic 1 only	-12.04	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1249	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00338	---
Signal to Noise Ratio	SNR		-49.41	dB
Effective Number of Bits	ENOB		7.92	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00232	---
Spurious Free Dynamic Range	SFDR		-52.71	dB

Table 1. Inputs (bold) and results of numerical example 1.

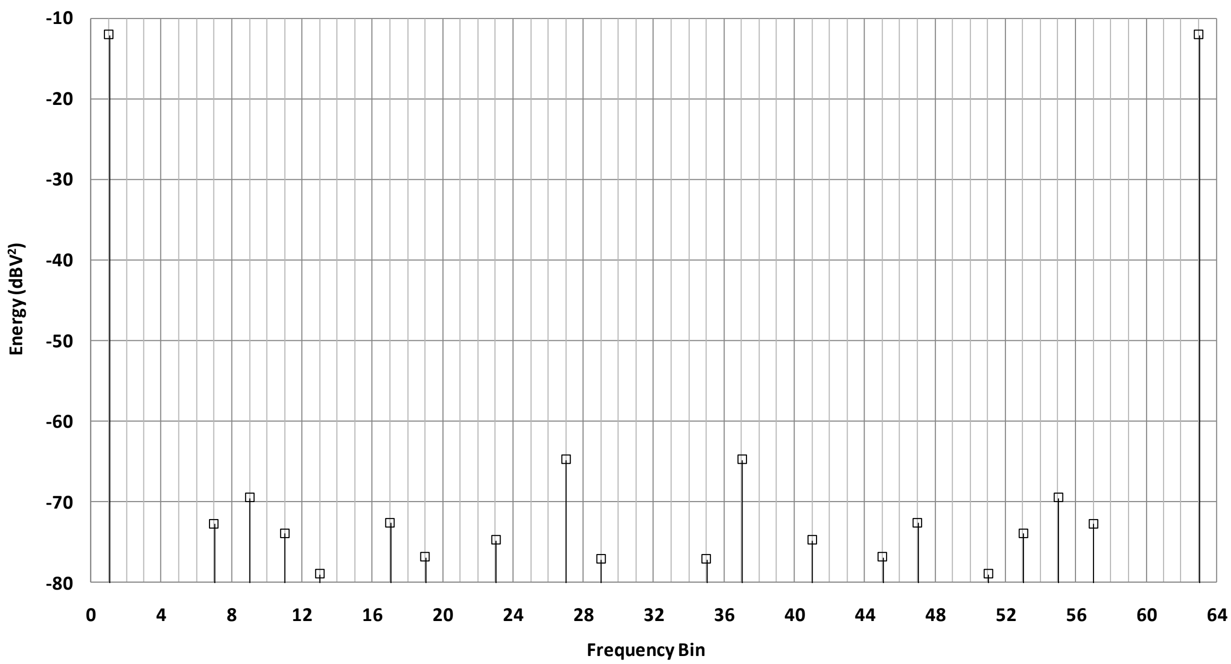

Figure 5. Spectrum of numerical example 1.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.1000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.5**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.124997	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.2500	V^2
Energy, Signal		Harmonic 1 only	-12.04	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1250	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00297	---
Signal to Noise Ratio	SNR		-50.54	dB
Effective Number of Bits	ENOB		8.10	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00171	---
Spurious Free Dynamic Range	SFDR		-55.33	dB

Table 2. Inputs (bold) and results of numerical example 2. Blue inputs differ from those in example 1.

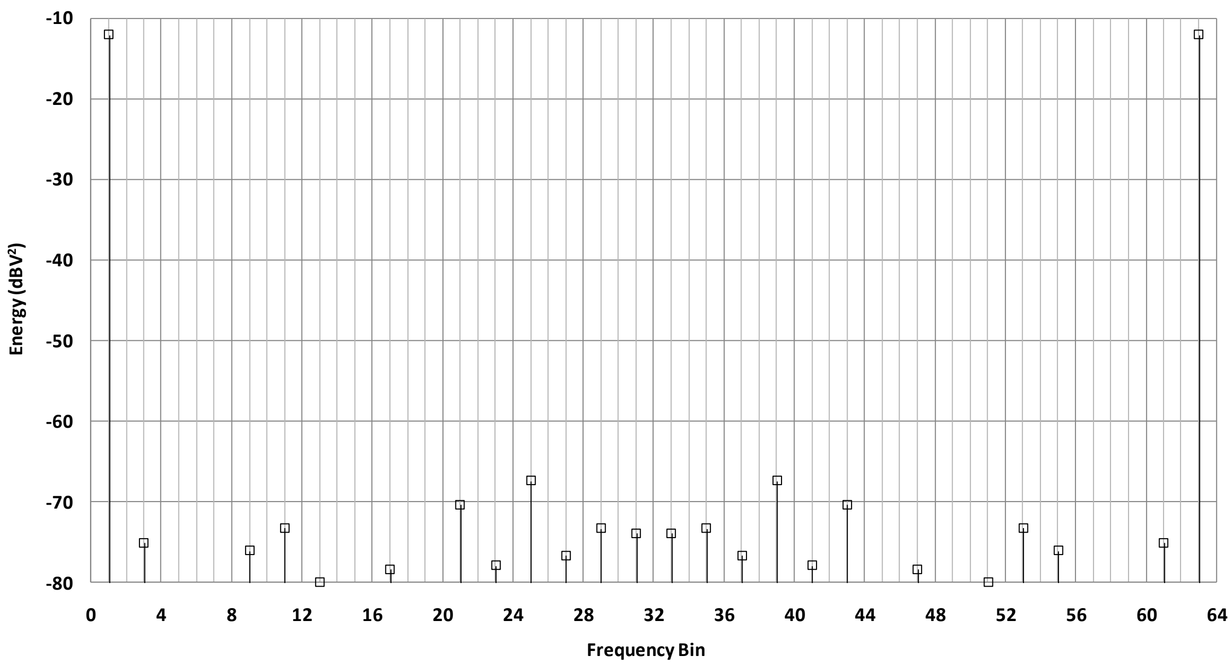

Figure 6. Spectrum of numerical example 2.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.03125**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.000484	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.0155	V^2
Energy, Signal		Harmonic 1 only	-36.17	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0005	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.05454	---
Signal to Noise Ratio	SNR		-25.27	dB
Effective Number of Bits	ENOB		3.90	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.02524	---
Spurious Free Dynamic Range	SFDR		-31.96	dB

Table 3. Inputs (bold) and results of numerical example 3. Blue inputs differ from those in example 1.

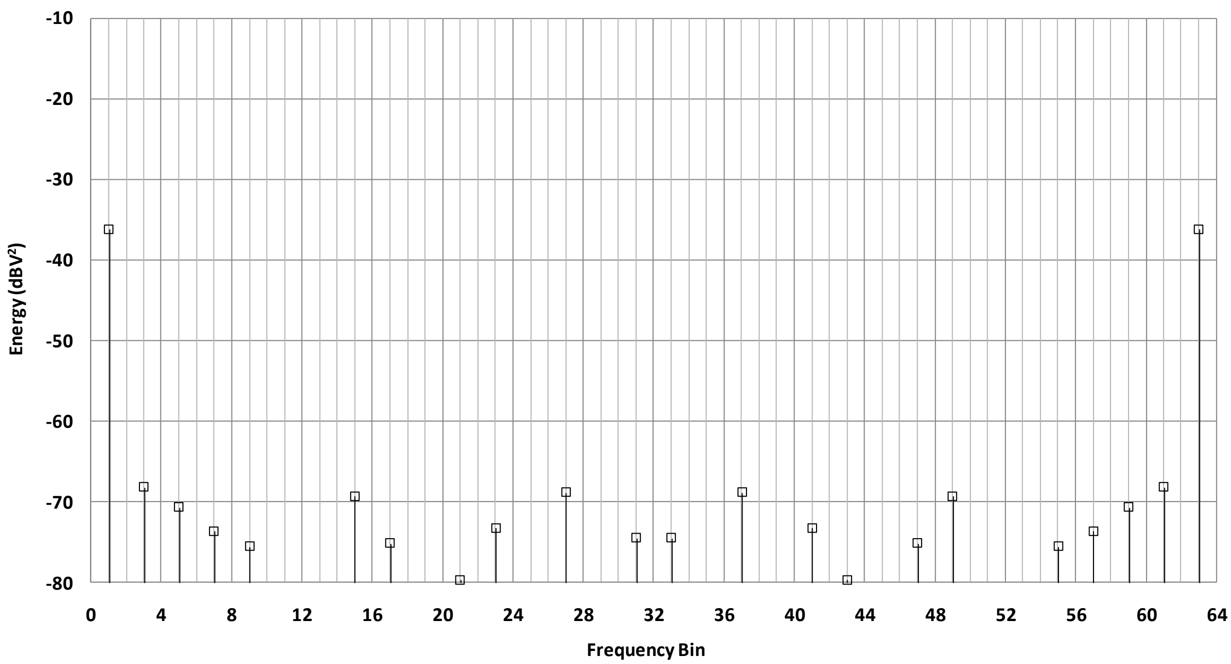

Figure 7. Spectrum of numerical example 3.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.5**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**16**	---
DAC Resolution Bits	DAC bits	Calculated, Q = $2^{(DAC\ bits)}$	4	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.062500	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.46875	V
Energy, Total		Average Amplitude Squared	0.124023	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.2487	V^2
Energy, Signal		Harmonic 1 only	-12.09	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1237	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00037	V^2
Energy, Error Max			0.00004	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.05454	---
Signal to Noise Ratio	SNR		-25.27	dB
Effective Number of Bits	ENOB		3.90	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.02524	---
Spurious Free Dynamic Range	SFDR		-31.96	dB

Table 4. Inputs (bold) and results of numerical example 4. Blue inputs differ from those in example 1.

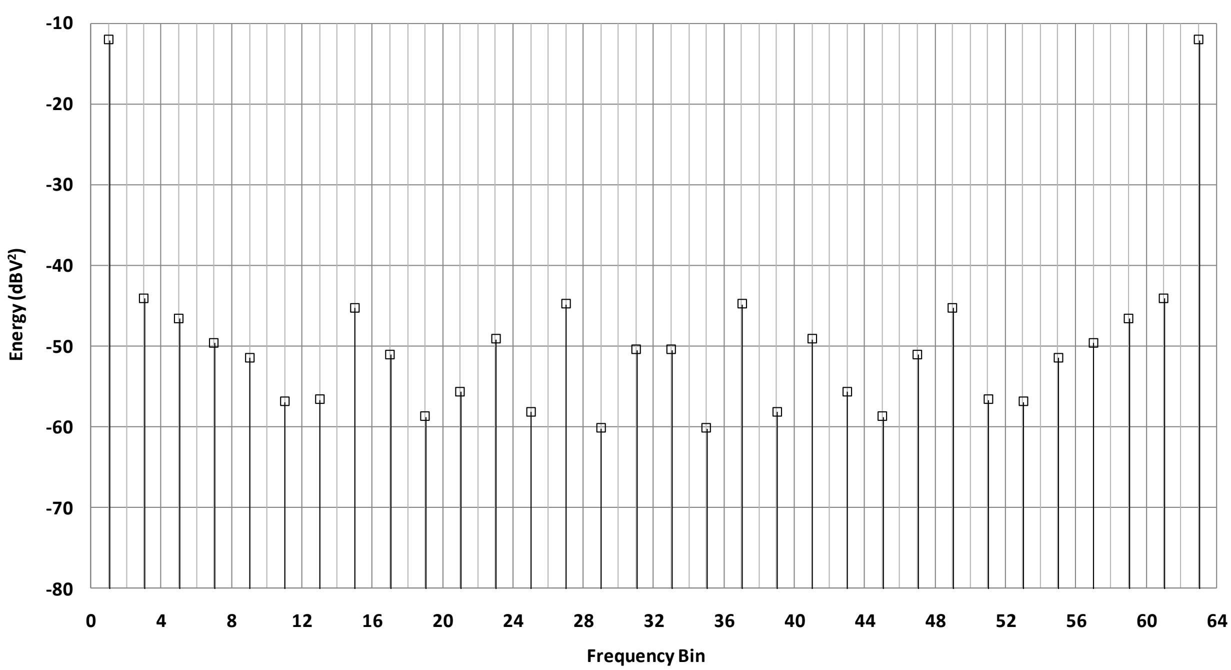

Figure 8. Spectrum of numerical example 4.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.5**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**2**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	1	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.500000	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.25000	V
Energy, Total		Average Amplitude Squared	0.062500	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.1592	V^2
Energy, Signal		Harmonic 1 only	-15.96	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0507	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.01180	V^2
Energy, Error Max			0.00283	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.48240	---
Signal to Noise Ratio	SNR		-6.33	dB
Effective Number of Bits	ENOB		0.76	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.33441	---
Spurious Free Dynamic Range	SFDR		-9.51	dB

Table 5. Inputs (bold) and results of numerical example 5. Blue inputs differ from those in example 1.

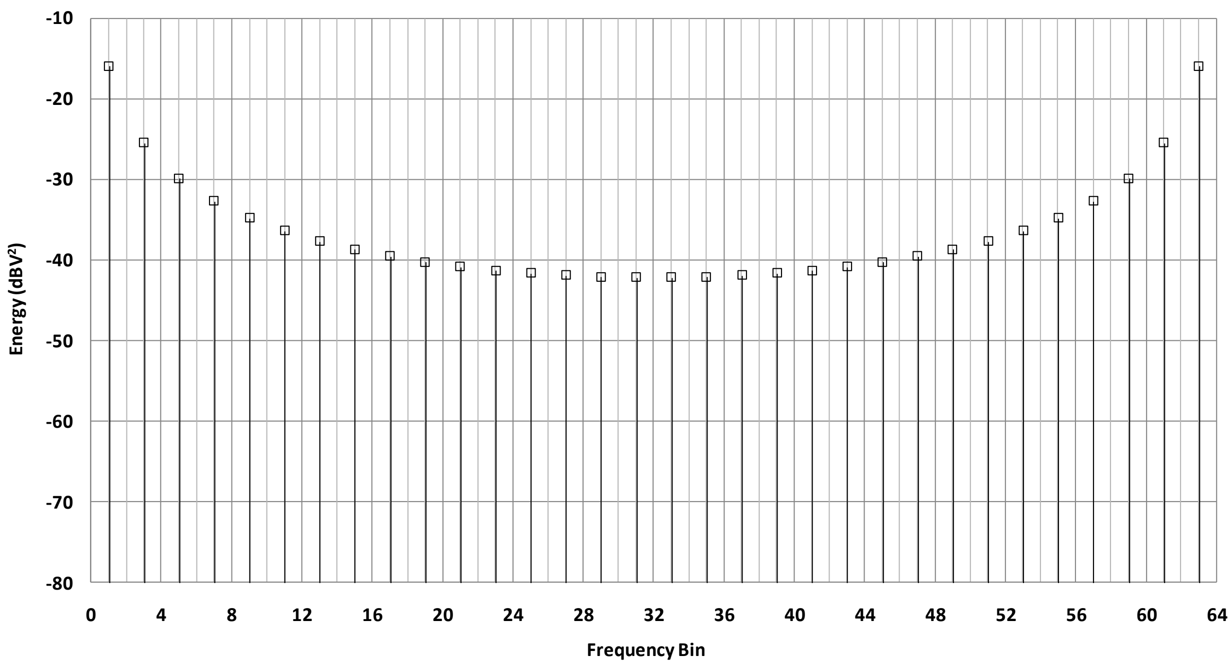

Figure 9. Spectrum of numerical example 5.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, ..., N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.525**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.134176	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.2590	V^2
Energy, Signal		Harmonic 1 only	-11.74	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1341	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00006	V^2
Energy, Error Max			0.00001	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.02026	---
Signal to Noise Ratio	SNR		-33.87	dB
Effective Number of Bits	ENOB		5.33	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.01374	---
Spurious Free Dynamic Range	SFDR		-37.24	dB

Table 6. Inputs (bold) and results of numerical example 6. Blue inputs differ from those in example 1.

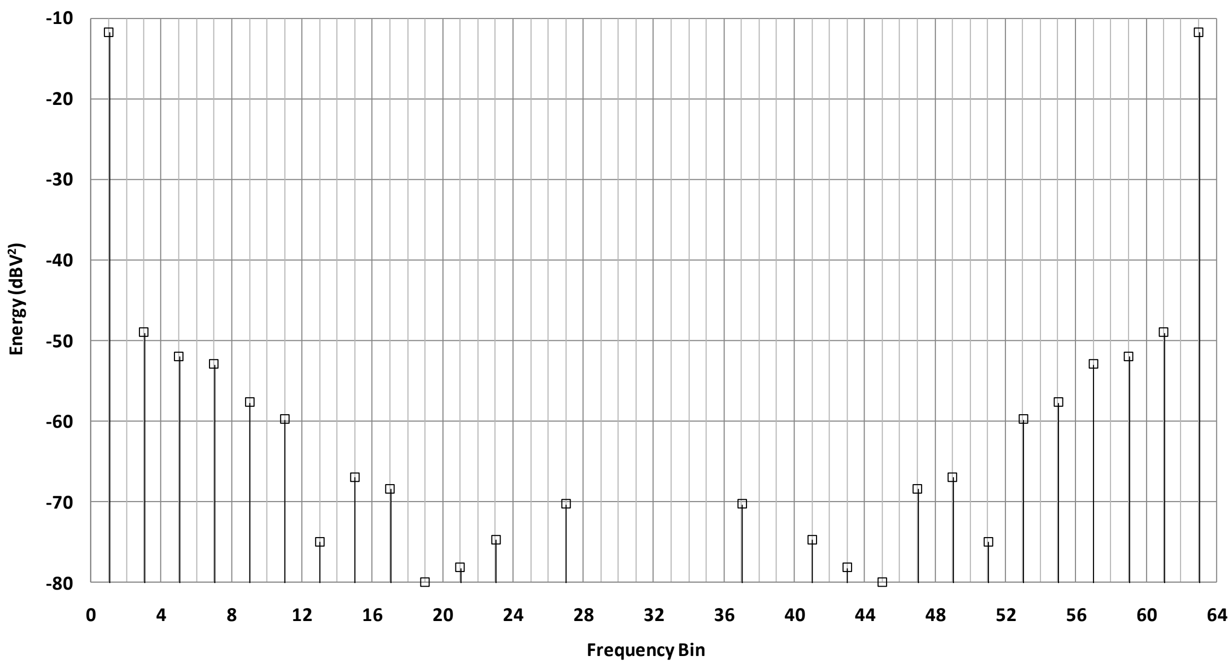

Figure 10. Spectrum of numerical example 6.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, ..., N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0625**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.5**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.118468	V^2
Energy, DC		Harmonic 0	0.00308491	V^2
Amplitude, Signal		Harmonic 1 only	0.2432	V^2
Energy, Signal		Harmonic 1 only	-12.28	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1182	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00022	V^2
Energy, Error Max			0.00004	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.04300	---
Signal to Noise Ratio	SNR		-27.33	dB
Effective Number of Bits	ENOB		4.25	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.02574	---
Spurious Free Dynamic Range	SFDR		-31.79	dB

Table 7. Inputs (bold) and results of numerical example 7. Blue inputs differ from those in example 1.

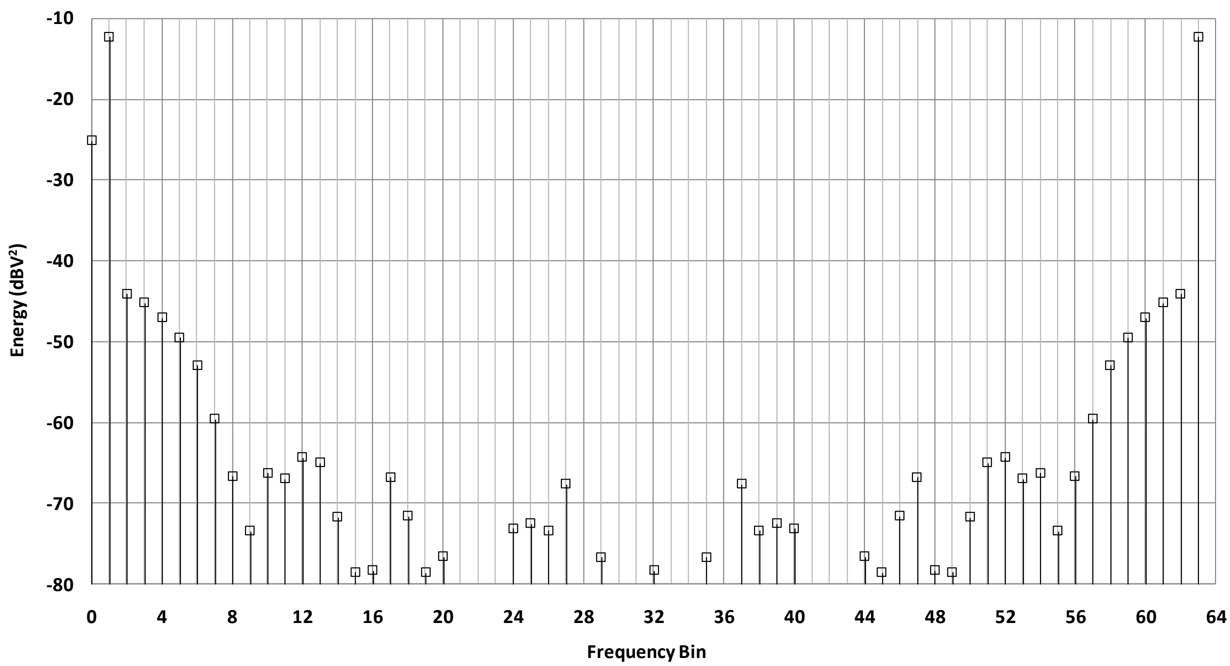

Figure 11. Spectrum of numerical example 7.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, ..., N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0625**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, Q = $2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, -(q-1/2)*Δ_{out}	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.095715	V^2
Energy, DC		Harmonic 0	0.00390625	V^2
Amplitude, Signal		Harmonic 1 only	0.2188	V^2
Energy, Signal		Harmonic 1 only	-13.20	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0957	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00403	---
Signal to Noise Ratio	SNR		-47.89	dB
Effective Number of Bits	ENOB		7.66	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00204	---
Spurious Free Dynamic Range	SFDR		-53.82	dB

Table 8. Inputs (bold) and results of numerical example 8. Blue inputs differ from those in example 1.

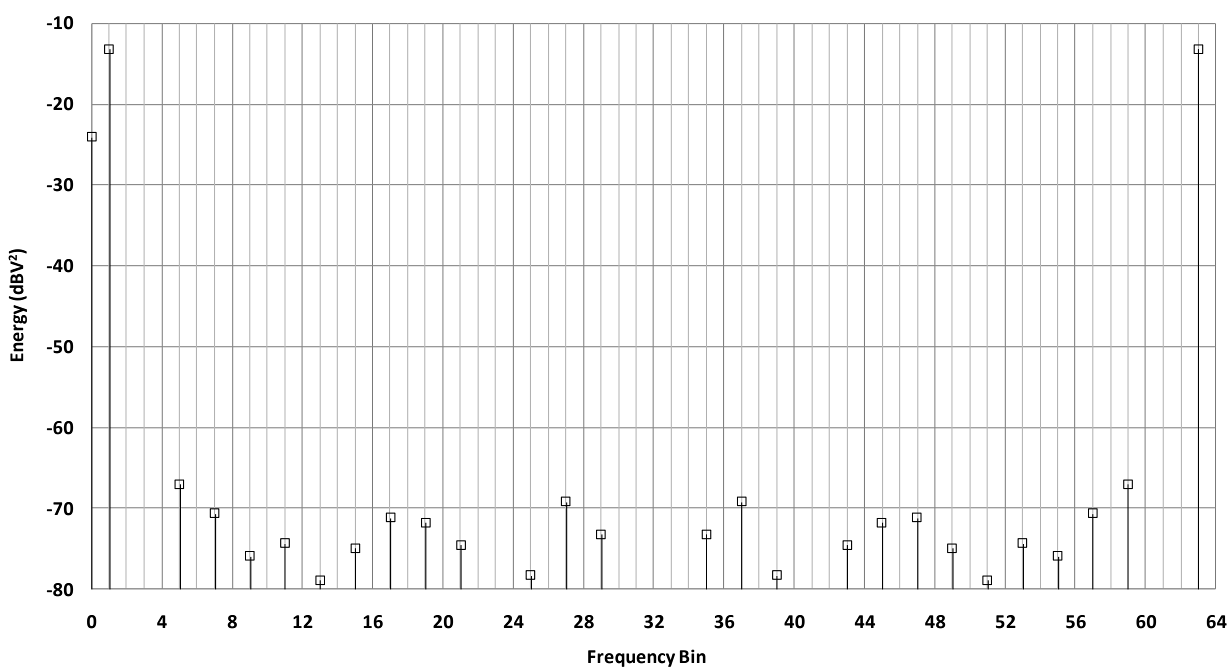

Figure 12. Spectrum of numerical example 8.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0000**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, Q = $2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, -(q-1/2)*Δ_{out}	Nominal w/o DNL	-0.49805	V
Energy, Total		Average Amplitude Squared	0.095715	V^2
Energy, DC		Harmonic 0	0	V^2
Amplitude, Signal		Harmonic 1 only	0.2188	V^2
Energy, Signal		Harmonic 1 only	-13.20	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0957	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00403	---
Signal to Noise Ratio	SNR		-47.89	dB
Effective Number of Bits	ENOB		7.66	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00204	---
Spurious Free Dynamic Range	SFDR		-53.82	dB

Table 9. Inputs (bold) and results of numerical example 9. Blue inputs differ from those in example 1.

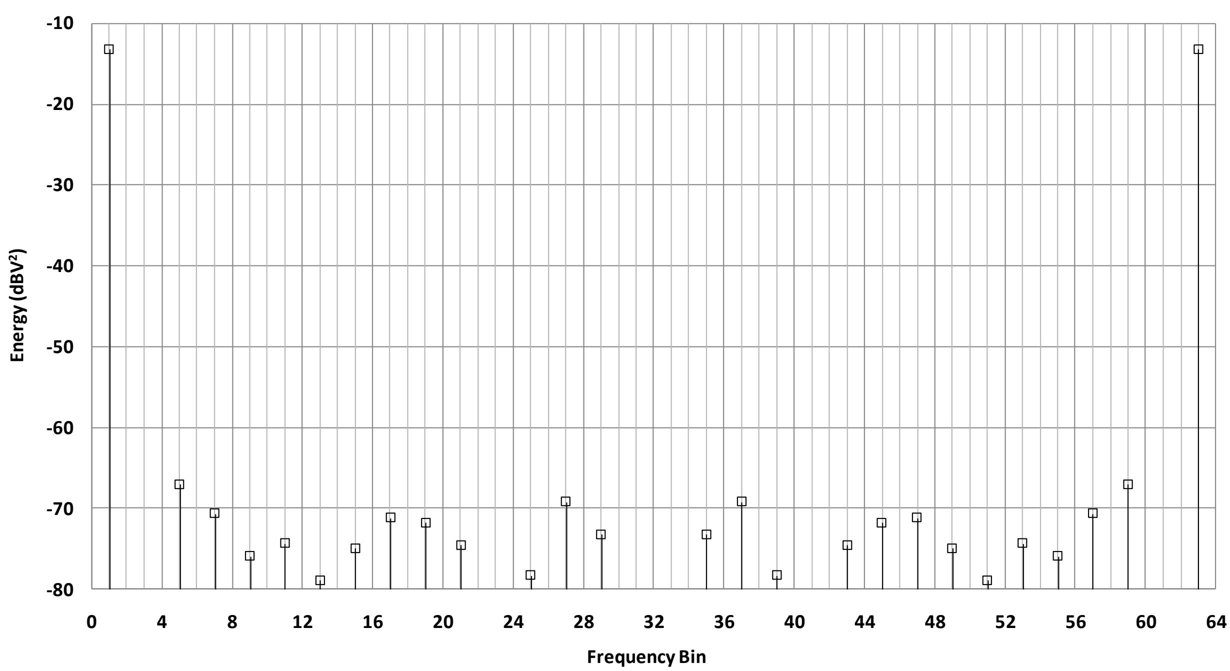

Figure 13. Spectrum of numerical example 9.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, ..., N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0000**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
DAC DNL Amplitude	A_{DNL}	Uniform random distributed DNL over codes in [$-A_{DNL}$,A_{DNL}]	**0.125**	LSB
Energy, Total		Average Amplitude Squared	0.097658	V^2
Energy, DC		Harmonic 0	2.5652E-05	V^2
Amplitude, Signal		Harmonic 1 only	0.2210	V^2
Energy, Signal		Harmonic 1 only	-13.11	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0977	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00435	---
Signal to Noise Ratio	SNR		-47.24	dB
Effective Number of Bits	ENOB		7.55	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00193	---
Spurious Free Dynamic Range	SFDR		-54.28	dB

Table 10. Inputs (bold) and results of numerical example 10. Blue inputs differ from those in example 1.

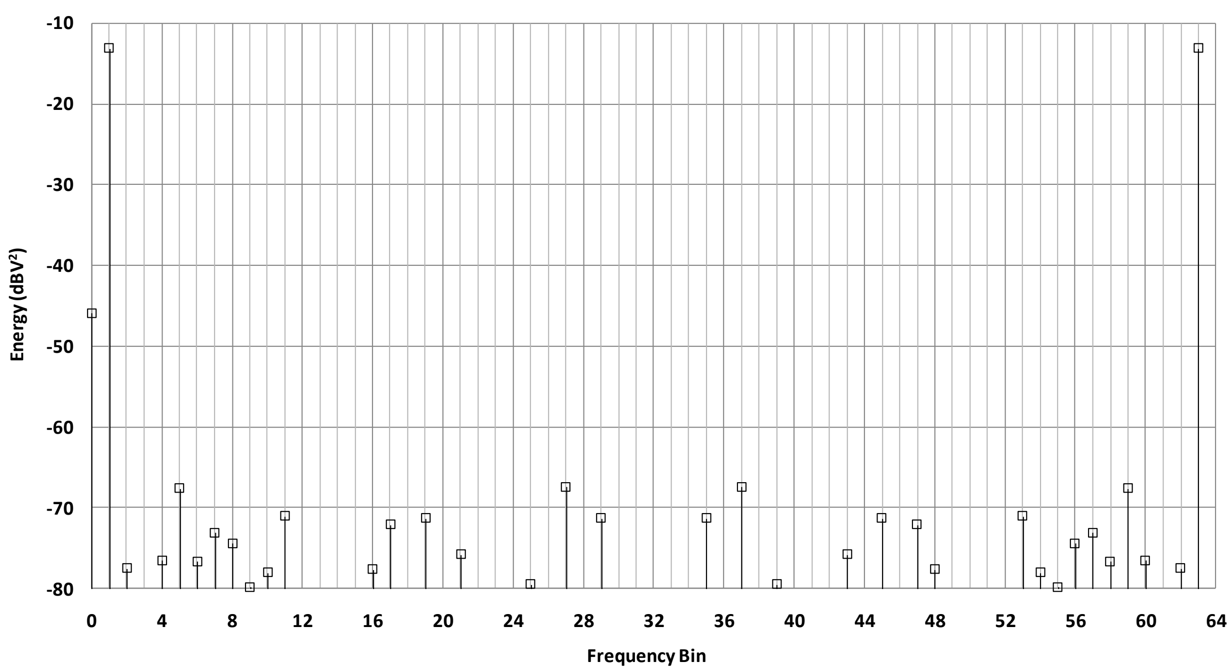

Figure 14. Spectrum of numerical example 10.

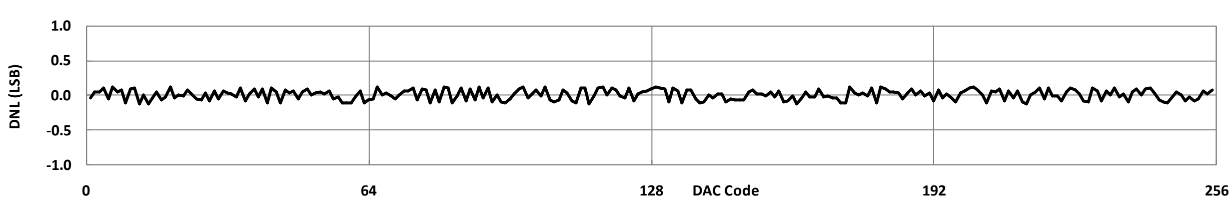

Figure 15. DNL for example 10.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, ..., N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0000**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, Q = $2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, -(q-1/2)*Δ_{out}	Nominal w/o DNL	-0.49805	V
DAC DNL Amplitude	A_{DNL}	Uniform random distributed DNL over codes in [-A_{DNL},A_{DNL}]	**0.250**	LSB
Energy, Total		Average Amplitude Squared	0.099622	V^2
Energy, DC		Harmonic 0	0.00010261	V^2
Amplitude, Signal		Harmonic 1 only	0.2232	V^2
Energy, Signal		Harmonic 1 only	-13.03	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.0996	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00000	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00563	---
Signal to Noise Ratio	SNR		-44.99	dB
Effective Number of Bits	ENOB		7.18	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00227	---
Spurious Free Dynamic Range	SFDR		-52.90	dB

Table 11. Inputs (bold) and results of numerical example 11. Blue inputs differ from those in example 1.

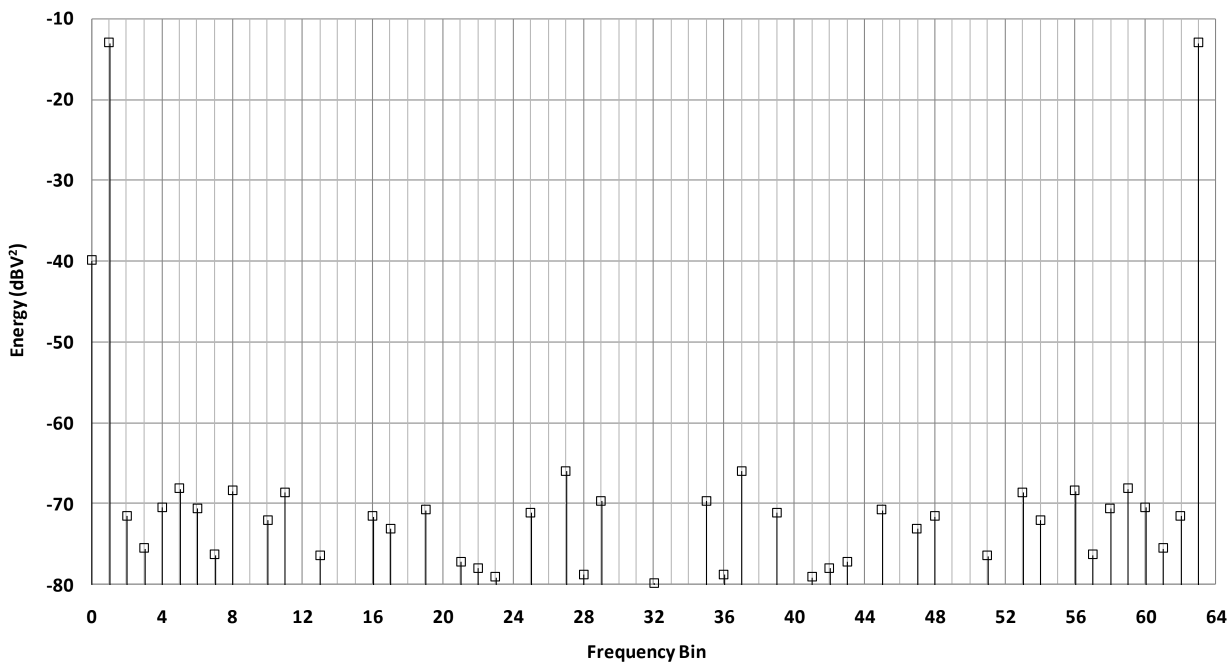

Figure 16. Spectrum of numerical example 11.

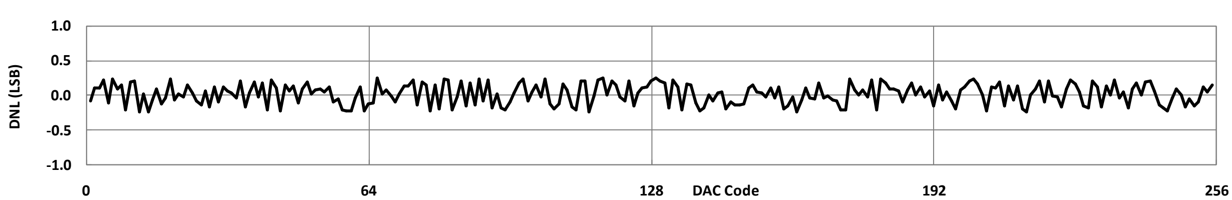

Figure 17. DNL for example 11.

Parameter	Symbol	Condition	Value	Unit
Number of bins	**N**		**64**	—
Input Signal Phase	**ϕ_0**	**In cosine signal description**	**0.0000**	**rad**
Input Signal Frequency	**p_{in}**	**Pick 1, ..., N-1, excluding N/2**	**1**	**bin**
Input Signal Offset	**x_0'**	**Relative to DAC input full scale range (FSR_{in})**	**0.0000**	—
Input Signal Amplitude	**A'**	**Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input**	**0.4375**	—
DAC Output Full Scale Range	**FSR_{out}**		**1**	**V**
DAC Resolution Levels	**Q, 2*q**	**Even integer greater than zero**	**256**	—
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	—
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
DAC DNL Amplitude	**A_{DNL}**	**Uniform random distributed DNL over codes in [$-A_{DNL}$,A_{DNL}]**	**0.500**	**LSB**
Energy, Total		Average Amplitude Squared	0.103611	V^2
Energy, DC		Harmonic 0	0.00041043	V^2
Amplitude, Signal		Harmonic 1 only	0.2276	V^2
Energy, Signal		Harmonic 1 only	-12.86	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1036	V^2
Energy, Error		Harmonic 2, 3, 4, ..., N-2	0.00001	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.00924	—
Signal to Noise Ratio	SNR		-40.68	dB
Effective Number of Bits	ENOB		6.47	—
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00335	—
Spurious Free Dynamic Range	SFDR		-49.50	dB

Table 12. Inputs (bold) and results of numerical example 12. Blue inputs differ from those in example 1.

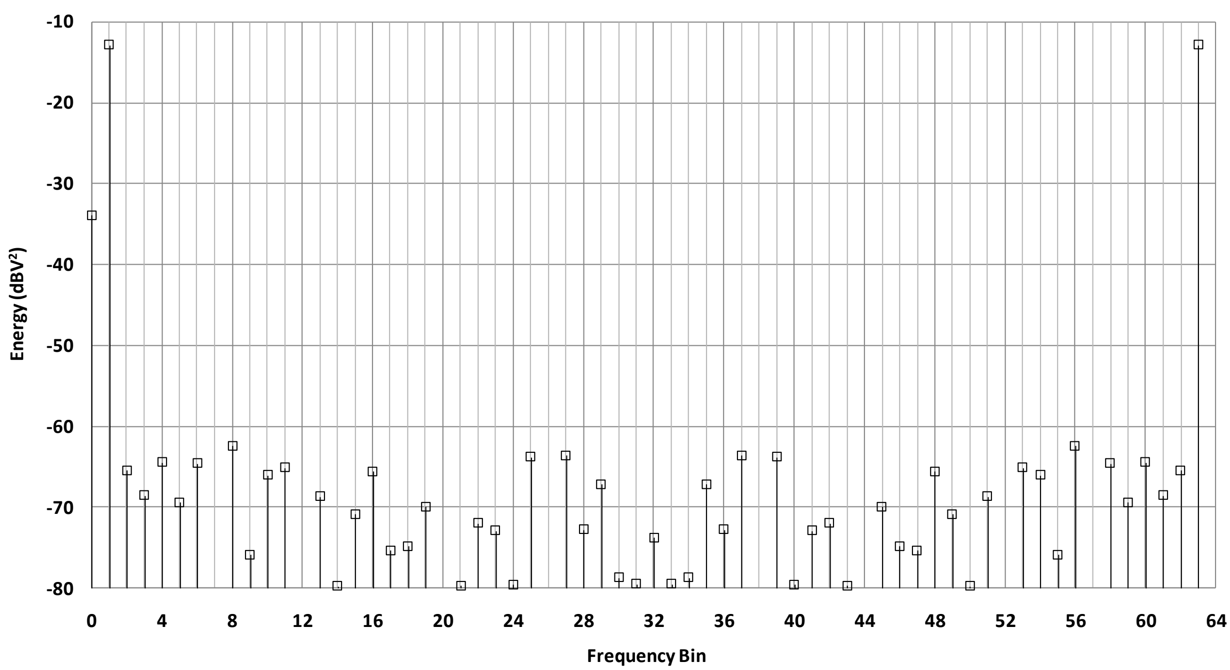

Figure 18. Spectrum of numerical example 12.

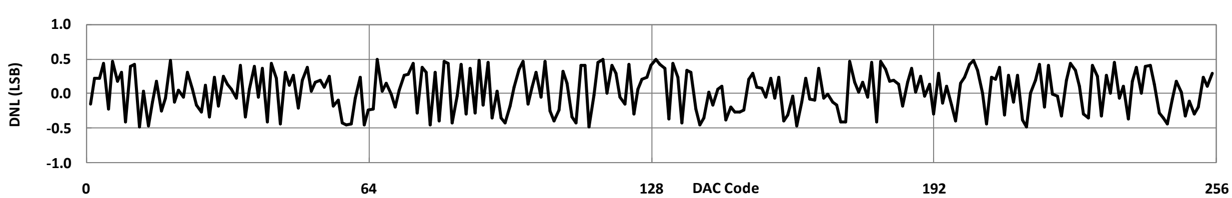

Figure 19. DNL for example 12.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0000**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, Q = $2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
DAC DNL Amplitude	A_{DNL}	Uniform random distributed DNL over codes in [$-A_{DNL}$,A_{DNL}]	**1.000**	LSB
Energy, Total		Average Amplitude Squared	0.111837	V^2
Energy, DC		Harmonic 0	0.00164171	V^2
Amplitude, Signal		Harmonic 1 only	0.2364	V^2
Energy, Signal		Harmonic 1 only	-12.53	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1118	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00003	V^2
Energy, Error Max			0.00000	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.01705	---
Signal to Noise Ratio	SNR		-35.37	dB
Effective Number of Bits	ENOB		5.58	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.00645	---
Spurious Free Dynamic Range	SFDR		-43.81	dB

Table 13. Inputs (bold) and results of numerical example 13. Blue inputs differ from those in example 1.

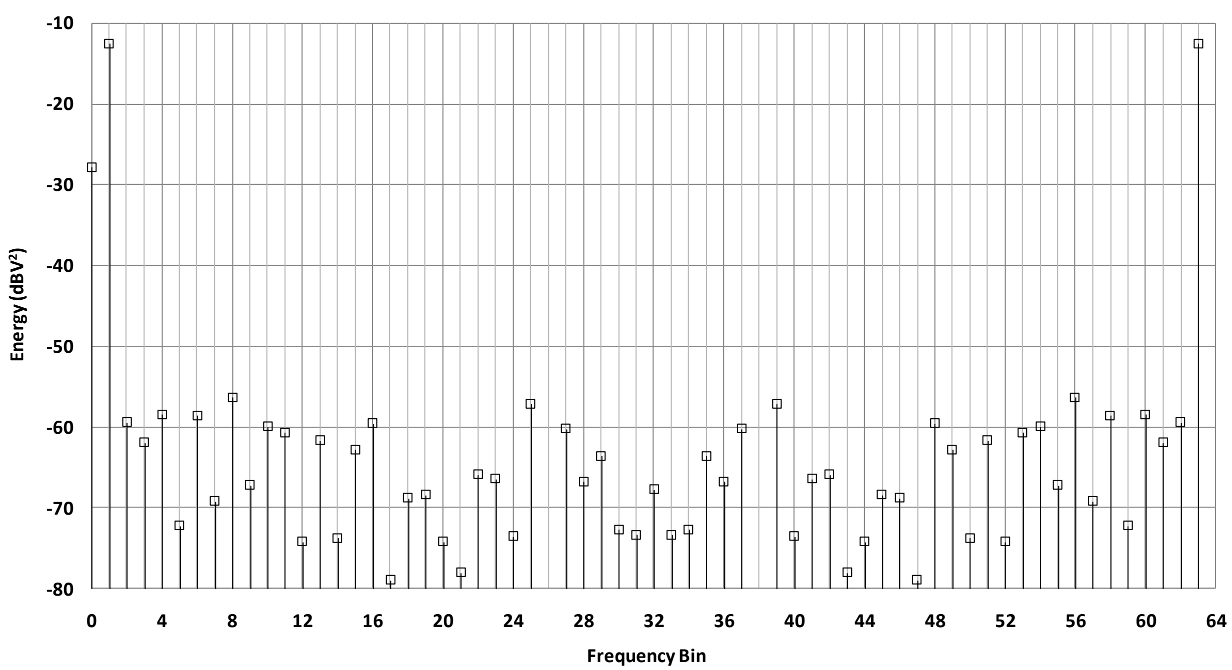

Figure 20. Spectrum of numerical example 13.

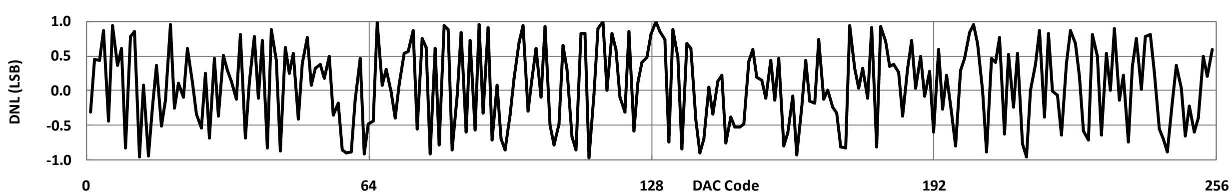

Figure 21. DNL for example 13.

Parameter	Symbol	Condition	Value	Unit
Number of bins	N		**64**	---
Input Signal Phase	ϕ_0	In cosine signal description	**0.0000**	rad
Input Signal Frequency	p_{in}	Pick 1, …, N-1, excluding N/2	**1**	bin
Input Signal Offset	x_0'	Relative to DAC input full scale range (FSR_{in})	**0.0000**	---
Input Signal Amplitude	A'	Relative to DAC input full scale range (FSR_{in}). 0.5 is full scale input	**0.4375**	---
DAC Output Full Scale Range	FSR_{out}		**1**	V
DAC Resolution Levels	Q, 2*q	Even integer greater than zero	**256**	---
DAC Resolution Bits	DAC bits	Calculated, $Q = 2^{(DAC\ bits)}$	8	---
DAC Step	Δ_{out}, LSB	Nominal w/o DNL	0.003906	V
DAC Lowest Output Level	o, L_0, $-(q-1/2)*\Delta_{out}$	Nominal w/o DNL	-0.49805	V
DAC DNL Amplitude	A_{DNL}	Uniform random distributed DNL over codes in [$-A_{DNL}$,A_{DNL}]	**2.000**	LSB
Energy, Total		Average Amplitude Squared	0.129275	V^2
Energy, DC		Harmonic 0	0.00656684	V^2
Amplitude, Signal		Harmonic 1 only	0.2541	V^2
Energy, Signal		Harmonic 1 only	-11.90	dBV^2
Energy, Signal		Harmonic 1 and N-1	0.1291	V^2
Energy, Error		Harmonic 2, 3, 4, …, N-2	0.00013	V^2
Energy, Error Max			0.00001	V^2
Signal to Noise Ratio	SNR	Aplitude ratio	0.03159	---
Signal to Noise Ratio	SNR		-30.01	dB
Effective Number of Bits	ENOB		4.69	---
Spurious Free Dynamic Range	SFDR	Aplitude ratio	0.01200	---
Spurious Free Dynamic Range	SFDR		-38.42	dB

Table 14. Inputs (bold) and results of numerical example 14. Blue inputs differ from those in example 1.

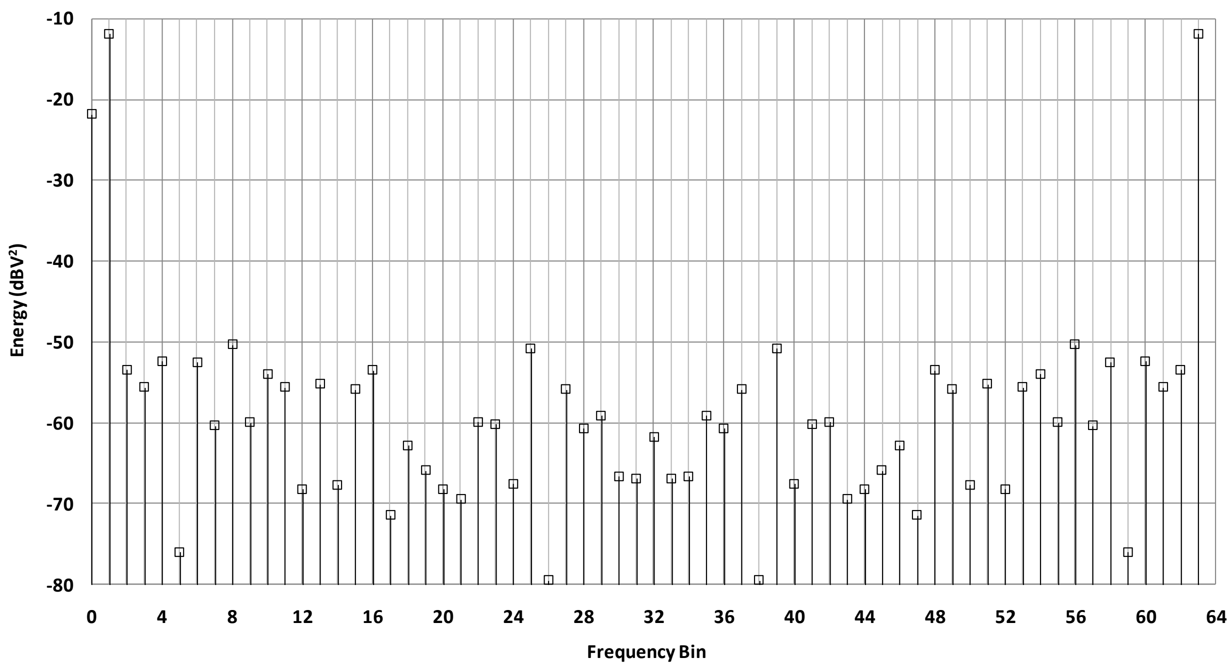

Figure 22. Spectrum of numerical example 14.

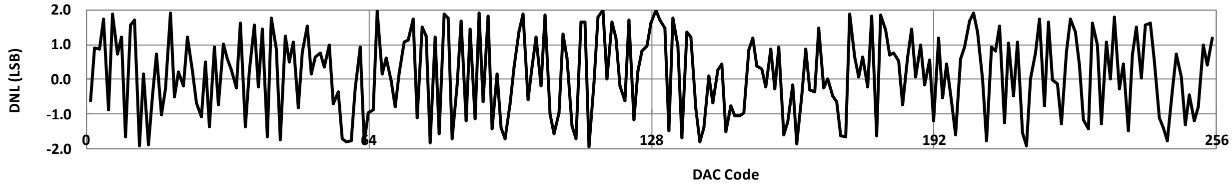

Figure 23. DNL for example 14.

Appendix A. Master Sum Formula

December 1998, Switzerland, on train to Basel

For any real a and any real x we have the "master sum formula"

$$\sum_{m=-\infty}^{\infty} \frac{1^{(a+m)x}}{a+m} = \frac{\pi}{\sin(\pi a)} 1^{aMR(x)} \tag{39}$$

or the equivalent form

$$\sum_{m=-\infty}^{\infty} \frac{1^{(h+mN)x}}{h+mN} = \frac{\frac{\pi}{N}}{\sin(\frac{\pi h}{N})} 1^{\frac{h}{N}MR(Nx)} \tag{40}$$

which is in turn equivalent to (27).

The master sum formula can be proven from equations 5 and 8 on p. 39 of [6]. A direct derivation using complex contour integration is presented here.

For any real a and any real ϕ we have

$$\begin{aligned} \sum_{m>-a}^{\infty} \frac{e^{i(a+m)\phi}}{a+m} &= i\int_{i\infty}^{\phi} \left(\sum_{m>-a}^{\infty} e^{i(a+m)z} \right) dz = \\ &= i\int_{i\infty}^{\phi} \frac{e^{i(E(-a)+1+a)z}}{1-e^{iz}} dz = \\ &= i\int_{\phi}^{i\infty} \frac{e^{i(a-E(a))z}}{e^{iz}-1} dz \end{aligned} \tag{41}$$

where we used the property of the Entier function $E(a) + E(-a) + 1 = 0$. The interchange of summation and integration in the first step requires that the z-integration contour lies entirely in the upper half of the complex plane, i.e. it touches the real axis only at the point ϕ.

We also have

$$\begin{aligned} \sum_{m<-a}^{\infty} \frac{e^{i(a+m)\phi}}{a+m} &= i\int_{-i\infty}^{\phi} \left(\sum_{m<-a}^{\infty} e^{i(a+m)z} \right) dz = \\ &= i\int_{-i\infty}^{\phi} \frac{e^{-i(E(a)+1-a)z}}{1-e^{-iz}} dz = \\ &= i\int_{-i\infty}^{\phi} \frac{e^{i(a-E(a))z}}{e^{iz}-1} dz \end{aligned} \tag{42}$$

This time, the first step requires that the z-integration contour lies entirely in the lower half of the complex plane, again touching the real axis only at the point ϕ.

Combining (41) and (42) we obtain:

$$\sum_{m=-\infty}^{\infty} \frac{e^{i(a+m)\phi}}{a+m} = i \int_{\Gamma(\phi)} \frac{e^{i(a-E(a))z}}{e^{iz}-1} dz \tag{43}$$

where the contour $\Gamma(\phi)$ runs from $-i\infty$ to $i\infty$, crossing the real axis only at point ϕ. By moving the end points of $\Gamma(\phi)$ to the right (left is also possible), we end up with a sum over residues, at points $z = 2\pi(E(\phi/2\pi)+1)$, $2\pi(E(\phi/2\pi)+2)$, …, since the integral over contour sections at infinity is zero:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \frac{e^{i(a+m)\phi}}{a+m} &= -2\pi i \sum_{n=E(\phi/2\pi)+1}^{\infty} e^{2\pi i a n} = \\ &= -2\pi i e^{2\pi i a(E(\phi/2\pi)+1)} \frac{1}{1-e^{2\pi i a}} = \\ &= -2\pi i e^{2\pi i a MR(\phi/2\pi)} \frac{1}{-2i\sin(\pi a)} \\ &= \frac{\pi}{\sin(\pi a)} e^{2\pi i a MR(\phi/2\pi)} \end{aligned} \tag{44}$$

Writing $\phi = 2\pi x$, this derivation reads:

$$\sum_{m=-\infty}^{\infty} \frac{1^{(a+m)x}}{a+m} = -2\pi i \sum_{n=E(x)+1}^{\infty} 1^{an} = \frac{\pi}{\sin(\pi a)} 1^{aMR(x)} \tag{45}$$

Which is the sum formula (39).

Equation (39) is extremely powerful with many useful special cases. By taking the imaginary part and then letting a approach zero, it follows that:

$$\begin{aligned} MR(x) &= \mathrm{Im}\left[\sum_{m=-\infty}^{\infty} \frac{1^{mx}}{2\pi m}\right] = \\ &= x + \frac{1}{2\pi i} \sum_{m\neq 0}^{\infty} \frac{1^{mx}}{m} = x + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{m} \end{aligned} \tag{46}$$

As another example, setting $x = \pi$ in (39) yields:

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{a+m} = \frac{\pi}{\sin(\pi a)} \tag{47}$$

which may be derived from 1.445.8 of [6]. Now taking $a = ½$ gives:

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} \tag{48}$$

We must now fill in a gap: The above forms of the master sum formula hold only for *x* non-integer. Residue type integral evaluations require great care when the integration contour passes exactly through one of the residue pole locations. Here this happens whenever *x* is integer. Examination shows that (45) changes in general to:

$$\sum_{m=-\infty}^{\infty} \frac{1^{(a+m)x}}{a+m} = \overline{-2\pi i \sum_{n=-E(x)+1}^{\infty} 1^{an}}^{\,x=x-,x+} = \frac{\pi}{\sin(\pi a)} \overline{1^{aMR(x)}}^{\,x=x-,x+} \tag{49}$$

The averages are over values infinitesimally smaller and larger than *x* respectively. Minor further algebra finally gives:

$$\sum_{m=-\infty}^{\infty} \frac{1^{(a+m)x}}{a+m} = \frac{\pi}{\sin(\pi a)} \begin{cases} 1^{aMR(x)} & , x \text{ non-integer} \\ 1^{ax}\cos(\pi a) & , x \text{ integer} \end{cases} = \begin{cases} \dfrac{\pi}{\sin(\pi a)} 1^{aMR(x)} & , x \text{ non-integer} \\ \dfrac{\pi}{\tan(\pi a)} 1^{ax} & , x \text{ integer} \end{cases} \tag{50}$$

Now we can derive another known special case of the master sum formula. Setting $x = 0$ in (50) gives:

$$\sum_{m=-\infty}^{\infty} \frac{1}{a+m} = \frac{\pi}{\tan(\pi a)} \tag{51}$$

A more traditional derivation of this result uses:

$$\sum_{m=-\infty}^{\infty} \frac{1}{a+k} = \frac{1}{a} + \sum_{m=1}^{\infty} \left[\frac{1}{a+m} + \frac{1}{a-m} \right] = \frac{1}{a} - \sum_{m=1}^{\infty} \frac{2a}{m^2 - a^2} \tag{52}$$

With 1.445.6 of [6] with $x = 0$, $m = 0$, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{1}{\alpha+k} = \frac{1}{\alpha} - 2\alpha \left[\frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos(\pi\alpha)}{\alpha \sin(\pi\alpha)} \right] = \frac{\pi}{\tan(\pi\alpha)} \tag{53}$$

Identifying α as *a*, this is (51).

Appendix B. Linear Finite Resolution ADC Spectrum

1996 - 1997, Dallas, TX, Texas Instruments

Equation (30) can be applied to the case of an ADC. The input wave x_{in} is now the ADC analog input sinusoidal wave, and the output wave x_n are the discrete output codes. The threshold assumption $n\Delta_{in}$, $n = -q', -q'+1, \ldots, -1, 0, 1, \ldots, q'-1, q'$, implies a perfectly linear ADC. It should be clear that we could generalize this to a non-linear threshold function case to cover non-linear ADC transfer curves, but this is not the main topic of this paper. Assuming the lowest output code is 0 and the code step is 1, we have $\Delta_{out,n} = 1$, and:

$$x_f^D = \sum_{h=0}^{N-1} \frac{\delta_{(f, hf_{in}) \bmod (f_{clk})}}{2iN\sin(\frac{\pi h}{N})} \sum_{n=-q'}^{q'} \left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)} \right] \tag{54}$$

As remarked in Step 2 of the DAC Output Spectrum Derivation, the first sum is merely a "redistribution" and in particular simplifies if $p_{in} = f_{in}/f_{bin}$ and $N = f_{clk}/f_{bin}$ are relatively prime, i.e. $\mathrm{GCD}(p_{in},N) = 1$. In this case, (54) can be written as:

$$x^D_{hf_{in} \bmod f_{clk}} = \frac{1}{2iN\sin(\frac{\pi h}{N})} \sum_{n=-q'}^{q'} \left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_n)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_n)}{2\pi}\right)} \right], \ \mathrm{GCD}(f_{in}/f_{bin}, f_{clk}/f_{bin}) = 1 \tag{55}$$

Because our results are exact, this holds all the way down to a two-level, or one-bit, ADC, i.e. a clocked comparator. Indeed, then (55) degenerates to:

$$x^D_{hf_{in} \bmod f_{clk}} = \frac{1}{2iN\sin(\frac{\pi h}{N})} \left[1^{\frac{h}{N}MR\left(\frac{N(\phi_0+\alpha_0)}{2\pi}\right)} - 1^{\frac{h}{N}MR\left(\frac{N(\phi_0-\alpha_0)}{2\pi}\right)} \right] \tag{56}$$

where, from (23)

$$\alpha_0 = \cos^{-1}\left(\frac{-x_0}{A}\right) \tag{57}$$

If we further simplify (57) with $x_0 = 0$, $\alpha_0 = \pi/2$, we find

$$x^D_{hf_{in} \bmod f_{clk}} = \frac{\sin(\frac{\pi h}{2})}{N\sin(\frac{\pi h}{N})} 1^{\frac{h}{N}MR\left(\frac{N\phi_0}{2\pi}\right)} \tag{58}$$

As is well known, there are only odd harmonics. Even after the multiple successive specializations above, our result is still in a more general form than the typical statement for the spectrum of a single comparator. This gives an appreciation for the generality of our full ADC result (54), which itself was still a special case of the still more general DAC result.

Notes and References

[1] The notation “:f_{bin}” under the summation symbol indicates that frequency f steps by f_{bin}.

[2] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes in C, Cambridge University Press, Second Edition, 1992, Chapter 12. Like these authors, we absorb 2π in the exponent, which leads to more symmetrical equations with simpler pre-factors. Note however that their Fourier transform pair convention makes $H(f)$ in (12.0.1) (or H_n in (12.1.9)) the “weight” of frequency component $-f$ ($-n$) rather than f (n) as in our convention.

[3] The series x_n repeats with period N, but this does not matter to us.

[4] The function $x(t)$ repeats with period T, but this does not matter to us.

[5] A very accessible discussion of some of these can be found in [2].

[6] I. S. Gradsteyn and I. M. Ryzhik, A. Jeffrey Ed., Table of Integrals, Series, and Products. Academic Press, 5th ed., San Diego, 1994.